Curve Fitting

CE 311 K - Introduction to Computer Methods

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Nonlinear Regression

- Minimize the residual between the data points and the curve -- least-squares regression
  - Linear $y_i = a_0 + a_1 x_i$
  - Quadratic $y_i = a_0 + a_1 x_i + a_2 x_i^2$
  - …
  - Exponential (base *e*) $y_i = a e^{b x_i}$
  - Power (base *x*) $y_i = a x_i^b$
  - Saturation-Growth $y_i = a \frac{x_i}{b + x_i}$

Exponential Relationship

- If the relationship is an exponential function

$$y_i = ae^{bx_i}$$

- To make it linear, take logarithm of both side

$$\ln(y_i) = \ln(a) + bx_i \quad \Rightarrow \quad Y_i = A + bx_i$$

- Now it's a linear relation between *Y* (=ln(*y*)) and *x*

- Need to estimate the values of *A* (=ln(*a*)) and *b*

Power Relationship

- If the relationship is a power function

$$y_i = ax_i^{\,b}$$

- To make it linear, take logarithm of both side

$$\ln(y_i) = \ln(a) + b\ln(x_i) \quad \Rightarrow \quad Y_i = A + bX_i$$

- Now it's linear between *Y* (=ln(*y*)) and *X* (= ln(*x*))

- Need to estimate the values of *A* (=ln(*a*)) and *b*

Saturation-Growth Relationship

- If the relationship is a saturation-growth function

$$y_i = \frac{a x_i}{b + x_i}$$

- To make it linear, invert the equation

$$\frac{1}{y} = \frac{b}{a}\frac{1}{x} + \frac{1}{a} \quad \Rightarrow \quad Y_i = A + BX_i$$

- Now it's linear between *Y* (=1/*y*) and *X* (= 1/*x*)
- Need to estimate the values of *A* (=1/*a*) and *B* (=*b/a*)

Some Examples

- Quadratic curve $y = a_0 + a_1 x + a_2 x^2$
  - Flow rating curve:
    - *Q* = measured discharge,
    - *H* = stage (height) of water behind outlet

$$Q = a_0 + a_1 H + a_2 H^2$$

- Power curve $y = a x^b$
  - Sediment transport:
    - c = concentration of suspended sediment
    - q = river discharge

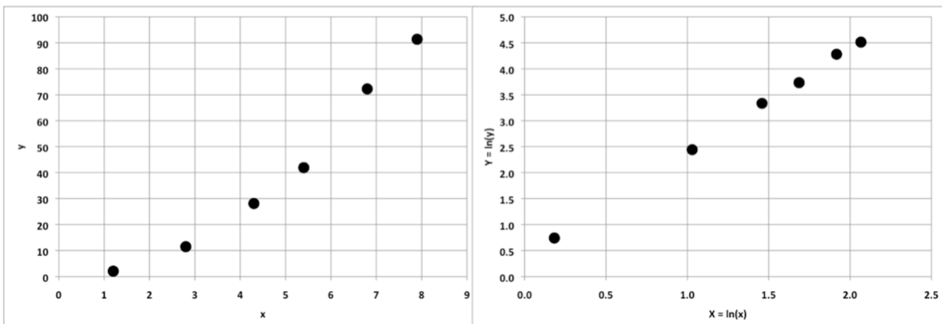
$$c = a q^b$$

  - Carbon adsorption:
    - *q* = mass of pollutant sorbed per unit mass of carbon,
    - *C* = concentration of pollutant *in* solution

$$q = K(c)^n$$

Example – Power Function

x	y	Log(x)	Log(y)
1.2	2.1	0.18	0.74
2.8	11.5	1.03	2.44
4.3	28.1	1.46	3.34
5.4	41.9	1.69	3.74
6.8	72.3	1.92	4.28
7.9	91.4	2.07	4.52



Example – Power Function

- Using the *log*'s, not the original *x*'s and *y*'s

$$\begin{bmatrix} n & \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} X_i & \sum_{i=1}^{n} X_i^2 \end{bmatrix} \begin{bmatrix} a \\ B \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} Y_i \\ \sum_{i=1}^{n} X_i Y_i \end{bmatrix}$$

$$\begin{bmatrix} 6 & 8.34 \\ 8.34 & 14.0 \end{bmatrix} \begin{bmatrix} a \\ B \end{bmatrix} = \begin{bmatrix} 19.1 \\ 31.4 \end{bmatrix}$$

$$\sum_{i=1}^{5} X_i = \sum_{i=1}^{5} \ln(x_i) = 8.34$$

$$\sum_{i=1}^{5} X_i^2 = \sum_{i=1}^{5} \ln(x_i)^2 = 14.0$$

$$\sum_{i=1}^{5} Y_i = \sum_{i=1}^{5} \ln(y_i) = 19.1$$

$$\sum_{i=1}^{5} X_i Y_i = \sum_{i=1}^{5} \ln(x_i)\ln(y_i) = 31.4$$