Solution of Linear Equations
2. Indirect Methods

CE 311 K - Introduction to Computer Methods

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Iterative Methods

- Consider the equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

- Rearrange
  - Unknowns on left
  - Knowns on right

$$x_1 = \frac{b_1 - (a_{12}x_2 + \cdots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + \cdots + a_{2n}x_n)}{a_{22}}$$

$$\vdots$$

$$x_n = \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn-1}x_{n-1})}{a_{nn}}$$

- Make a guess

$x_1^0, x_2^0, \cdots, x_n^0$

Jacobi Method

- Initial guess $x_1^0, x_2^0, \cdots, x_n^0$

- Next approximation of the solution

$$x_1^1 = \frac{b_1 - (a_{12}x_2^0 + \cdots + a_{1n}x_n^0)}{a_{11}}$$

$$x_2^1 = \frac{b_2 - (a_{21}x_1^0 + \cdots + a_{2n}x_n^0)}{a_{22}}$$

$$\vdots$$

$$x_n^1 = \frac{b_n - (a_{n1}x_1^0 + a_{n2}x_2^0 + \cdots + a_{nn-1}x_{n-1}^0)}{a_{nn}}$$

Jacobi Method

- After you do this *k* times (*k* iterations)

$$x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + \cdots + a_{1n}x_n^k)}{a_{11}}$$

$$x_2^{k+1} = \frac{b_2 - (a_{21}x_1^k + \cdots + a_{2n}x_n^k)}{a_{22}}$$

$$\vdots$$

$$x_n^{k+1} = \frac{b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \cdots + a_{nn-1}x_{n-1}^k)}{a_{nn}}$$

Example – Jacobi Method

- Equations

$$2x_1 - x_2 + x_3 = 5$$
$$x_1 + 3x_2 - 2x_3 = 8$$
$$x_1 + 2x_2 + 3x_3 = 10$$

- Rearrange

$$x_1^{k+1} = \frac{5 + x_2^k - x_3^k}{2}$$
$$x_2^{k+1} = \frac{8 - x_1^k + 2x_3^k}{3} \qquad k = 0,\ 1,\ 2,\ 3,\ \ldots$$
$$x_3^{k+1} = \frac{10 - x_1^k - 2x_2^k}{3}$$

$$x_1^{k=0} = 0,\ x_2^{k=0} = 0,\ x_3^{k=0} = 0$$

Example – Jacobi Method

- Initial guess $x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = 0$

$$x_1^{k=1} = \frac{5+0-0}{2} = 2.5 \qquad x_1^{k=2} = \frac{5 + 2.6667 - 3.3333}{2} = 2.1667$$
$$x_2^{k=1} = \frac{8-0+0}{3} = 2.66667 \qquad x_2^{k=2} = \frac{8 - 2.5 + 2(3.3333)}{3} = 4.0555$$
$$x_3^{k=1} = \frac{10-0-0}{3} = 3.3333 \qquad x_3^{k=2} = \frac{10 - 2.5 - 2(2.6667)}{3} = 0.7222$$

- After several iterations $x_1 = 3,\quad x_2 = 2,\quad x_3 = 1$

Errors and Stopping Criteria

- How do we know when to stop (k = 0,1,2,3,…?)
- Two major sources of error in numerical methods:
  - Roundoff error: Computers represent quantities with a finite number of digits
  - Truncation error: Numerical methods employ approximations to represent exact mathematical operations and quantities
- Consider the error between the numerical and analytical solutions
  - True value x
  - Approximate value $\tilde{x}$

Error Measures

- *E* = Error = True value - Approximation $E = x - \tilde{x}$
- *e* = relative error $e = \left| \frac{x - \tilde{x}}{x} \right|$
- *d* = significant digits $e = \left| \frac{x - \tilde{x}}{x} \right| < \frac{1}{2} 10^{-d}$

Example

- Approximate $\tilde{\pi} = 3.1416$
- "True" value $\pi = 3.1415927$
- Find: error, relative error, significant digits in approximation

$$
\begin{aligned}
E &= \pi - \tilde{\pi} \\
&= 3.1415927 - 3.1416 \\
&= -0.0000073
\end{aligned}
$$

$$
\begin{aligned}
e &= \left|\frac{\pi - \tilde{\pi}}{\pi}\right| = \left|\frac{-0.0000073}{3.1415927}\right| \\
&= 0.0000023237 \\
&= 2.3237x10^{-6}
\end{aligned}
$$

Example

- Find: significant digits in approximation

$$e < \frac{1}{2}10^{-d}$$

$$
\begin{aligned}
d &< -\frac{\ln(2e)}{\ln(10)} \\
&= -\frac{\ln(2 * 0.0000023237)}{\ln(10)} \\
&= 5.3329
\end{aligned}
$$

$$
\begin{aligned}
2e &< 10^{-d} \\
\frac{\ln(2e)}{\ln(10)} &< -d
\end{aligned}
$$

Approximate Error

- E_A is the approximate error between the current approximate value and our previous approximate value

$$e_A = \left| \frac{\tilde{x}^{k+1} - \tilde{x}^{k}}{\tilde{x}^{k+1}} \right|$$

Example

- Taylor Series for e^x $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
- Estimate e^x for $x = 0.5$
  - Start with 1 term and add more and more terms
  - Compute the value and the error after adding each new term
  - Add terms until the error $e < 0.005$

Example

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

- Using the first term

$$e^x \approx 1$$

- Using the first 2 terms

$$e^x \approx 1 + x = 1 + 0.5 = 1.5$$

- Using the first 3 terms

$$e^x \approx 1 + x + \frac{x^2}{2} = 1 + 0.5 + 0.125 = 1.625$$

Example

- From a calculator, e^x = 1.648721271

$$e = \left|\frac{x - \tilde{x}}{x}\right| = \left|\frac{1.648721271 - 1.5}{1.648721271}\right| = 0.09 = 9\%$$

# of Terms	Result, $e^{0.5}$	Error, e
1	1	0.393
2	1.5	0.09
3	1.625	0.014
4	1.645833333	0.0017
5	1.648437500	0.00017
6	1.648697917	0.000014

Jacobi Method

$$x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + \cdots + a_{1n}x_n^k)}{a_{11}}$$

$$x_2^{k+1} = \frac{b_2 - (a_{21}x_1^k + \cdots + a_{2n}x_n^k)}{a_{22}}$$

$$\vdots$$

$$x_n^{k+1} = \frac{b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \cdots + a_{nn-1}x_{n-1}^k)}{a_{nn}}$$

We stop when

$$e_1 = \left| \frac{\tilde{x}_1^{k+1} - \tilde{x}_1^k}{\tilde{x}_1^{k+1}} \right| < e$$

$$e_2 = \left| \frac{\tilde{x}_2^{k+1} - \tilde{x}_2^k}{\tilde{x}_2^{k+1}} \right| < e$$

$$\vdots$$

$$e_n = \left| \frac{\tilde{x}_n^{k+1} - \tilde{x}_n^k}{\tilde{x}_n^{k+1}} \right| < e$$

Gauss Seidel Method (better)

- Current approximation becomes available after each step – use it

$$x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + \cdots + a_{1n}x_n^k)}{a_{11}}$$

$$x_2^{k+1} = \frac{b_2 - (a_{21}x_1^{k+1} + \cdots + a_{2n}x_n^k)}{a_{22}}$$

$$\vdots$$

$$x_n^{k+1} = \frac{b_n - (a_{n1}x_1^{k+1} + a_{n2}x_2^{k+1} + \cdots + a_{nn-1}x_{n-1}^{k+1})}{a_{nn}}$$

Gauss-Seidel Method - Example

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 3x_2 - x_3 &= 8 \\ -x_2 + 2x_3 &= -5 \end{aligned}$$

convenient initial guess

$$x_1^0 = 0, x_2^0 = 0, x_3^0 = 0$$

$$x_1^{k+1} = \frac{1 + x_2^k}{2}$$

$$x_2^{k+1} = \frac{8 + x_1^{k+1} + x_3^k}{3}$$

$$x_3^{k+1} = \frac{-5 + x_2^{k+1}}{2}$$

after 1 iteration of the method

$$x_1^1 = \frac{1 + x_2^0}{2} = \frac{1+0}{2} = 0.5$$

$$x_2^1 = \frac{8 + x_1^1 + x_3^0}{3} = \frac{8 + 0.5 + 0}{3} = 2.8333$$

$$x_3^1 = \frac{-5 + x_2^1}{2} = \frac{-5 + 2.83333}{2} = -1.08333$$

Gauss-Seidel Method

after 1 iteration

$x_1^1 = 0.5$

$x_2^1 = 2.8333$

$x_3^1 = -1.08333$

after 2 iterations

$$x_1^2 = \frac{1 + x_2^1}{2} = \frac{1 + 0.5}{2} = 1.9167$$

$$x_2^2 = \frac{8 + x_1^2 + x_3^1}{3} = \frac{8 + 1.9167 + (-1.0833)}{3} = 2.9444$$

$$x_3^2 = \frac{-5 + x_2^2}{2} = \frac{-5 + 2.9444}{2} = -1.0278$$

after 9 iterations

$$x_1 = 2$$

$$x_2 = 3$$

$$x_3 = -1$$