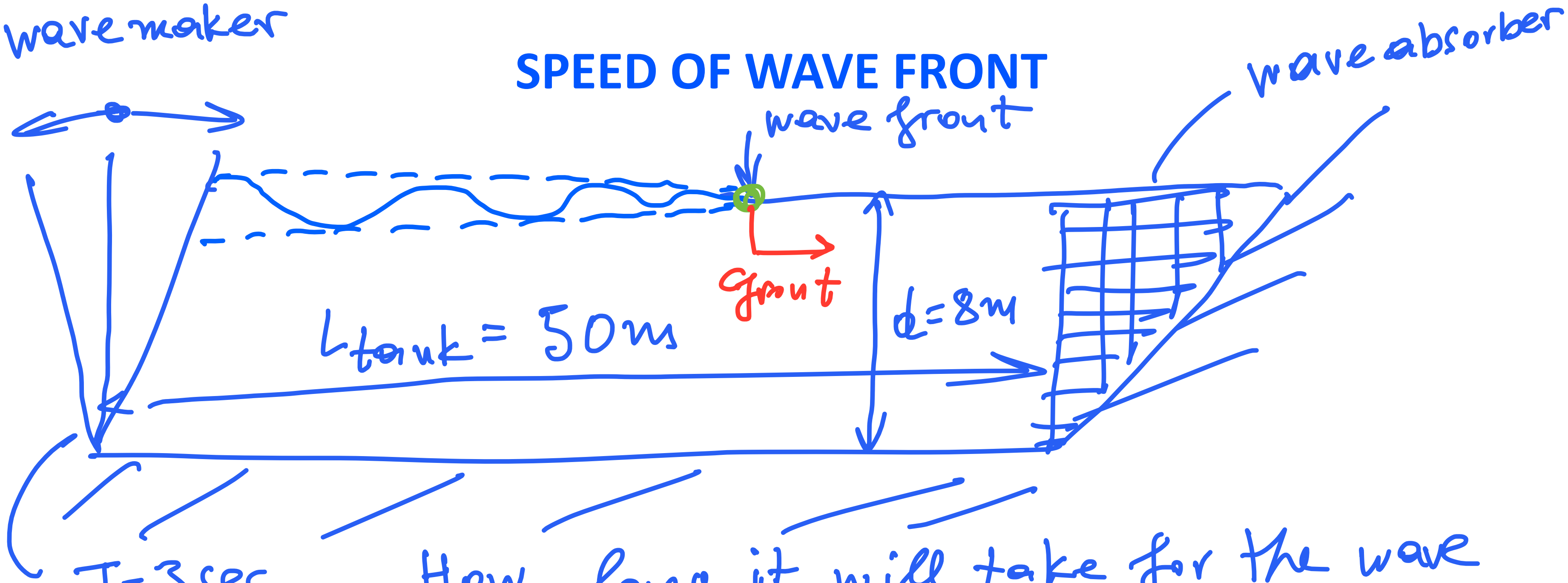
SPEED OF WAVE FRONT



T=3sec

How long it will take for the wave front to travel along the tank?

$$L_0 = \frac{gT^2}{2\pi} = 14.05\,m \qquad \frac{8}{14.05} = \frac{d}{L_0} > \frac{1}{2} \Rightarrow \underline{\text{deep } H_2O}$$

$$L = L_0$$

$$C = \frac{L}{T} = \frac{14.05}{3} = 4.68\,\frac{m}{s}$$

$$\underline{t_{cross}} = \frac{L_{tank}}{C} = \frac{50}{4.68} = \underline{10.7\,sec}$$

In reality the t_{cross} would be 2 times larger which means $\underline{C_{front} = \frac{C}{2}}$!!!

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2 WAVES WITH SLIGHLTY DIFFERENT LENGTH/PERIOD

deep H_2O

① $T_1 = 10\,sec$ $\quad H_1 = 2ft$

$$L_1 = \frac{g T_1^2}{2\pi} = 512.5\,ft$$

$$\eta_1 = \frac{H_1}{2}\cos(k_1 x - \omega_1 t)$$

② $L_2 = 1.05 \times L_1 = 538.1\,ft$ $\quad H_2 = 2ft$

$$L_2 = \frac{g T_2^2}{2\pi} \rightsquigarrow T_2 = \sqrt{\frac{2\pi L_2}{g}} = 10.25\,sec$$

$$\eta_2 = \frac{H_2}{2}\cos(k_2 x - \omega_2 t)$$

$$\underline{k_1} = \frac{2\pi}{L_1} = \underline{0.01226\,ft^{-1}}\ ;\quad \underline{\omega_1} = \frac{2\pi}{T_1} = \underline{0.628\,sec^{-1}}$$

$$\underline{k_2} = \frac{2\pi}{L_2} = \underline{0.01168\,ft^{-1}}\ ;\quad \underline{\omega_2} = \frac{2\pi}{T_2} = \underline{0.613\,sec^{-1}}$$

Compound wave: $\eta = \eta_1 + \eta_2$ $\quad (H_1 = H_2 = 2ft)$

$$\eta = \frac{H}{2}\left[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)\right]$$

From trig: $\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

GROUP OF 2 WAVES WITH SLIGHLTY DIFFERENT L or T (DEEP WATER)

$$\eta=\frac{H}{\cancel{2}}\cancel{2}\cos\Big[\underbrace{\overbrace{\frac{k_1+k_2}{2}}^{k_p}x-\overbrace{\frac{\omega_1+\omega_2}{2}}^{\omega_p}t}_{\frac{\alpha+\beta}{2}}\Big]\cos\Big[\underbrace{\overbrace{\frac{k_1-k_2}{2}}^{k_m}x-\overbrace{\frac{\omega_1-\omega_2}{2}}^{\omega_m}t}_{\frac{\alpha-\beta}{2}}\Big]$$

$\alpha=k_1x-\omega_1t$

$\beta=k_2x-\omega_2t$

$$k_p=\frac{k_1+k_2}{2}=0.01197\ ft^{-1}$$

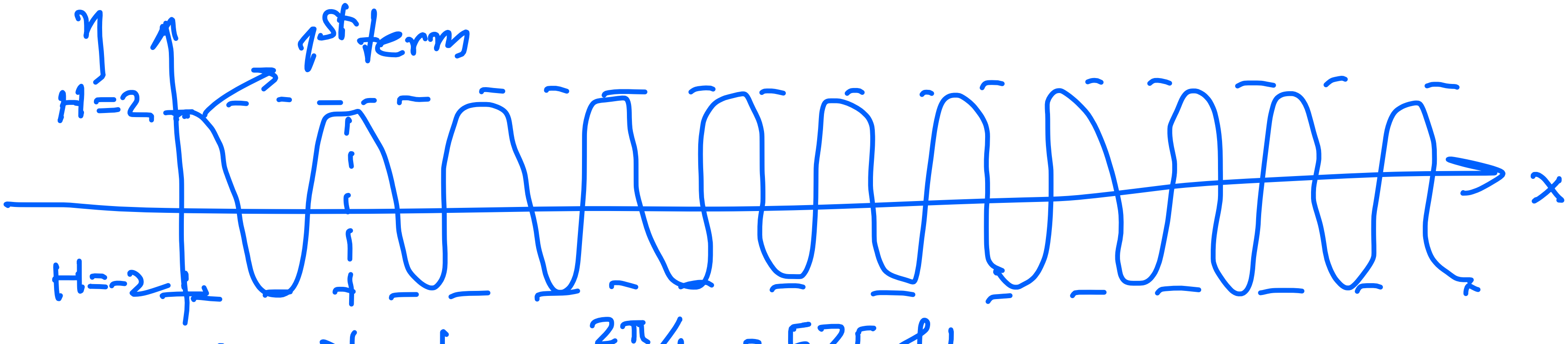
$$\omega_p=\frac{\omega_1+\omega_2}{2}=0.6205\ sec^{-1}$$

$$k_m=\frac{k_1-k_2}{2}=0.00029\ ft^{-1}$$

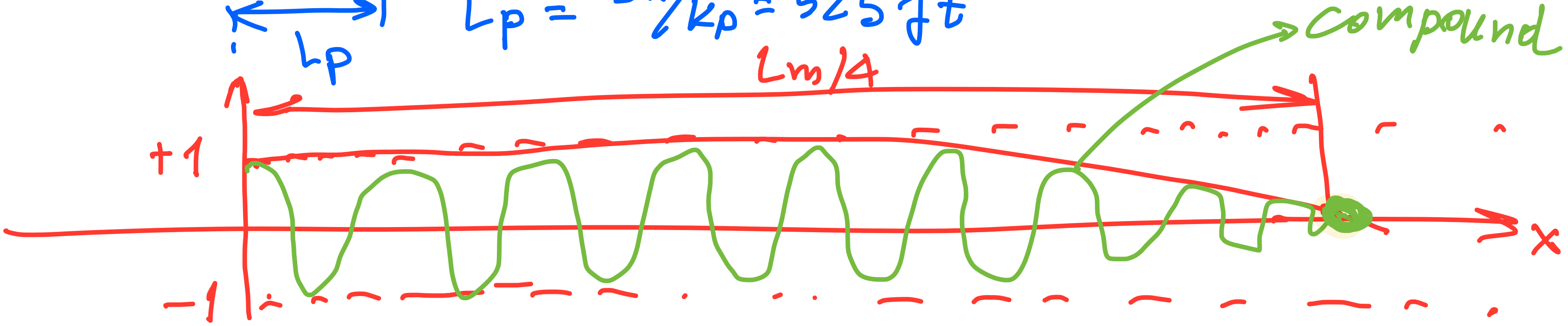
$$\omega_m=\frac{\omega_1-\omega_2}{2}=0.0075\ sec^{-1}$$

"small" values!

$$\eta=\underbrace{H\cos(k_px-\omega_pt)}_{1^{st}\ term}\underbrace{\cos(k_mx-\omega_mt)}_{2^{nd}\ term}$$



$L_p = 2\pi/k_p = 525$ ft



$$L_m = \frac{2\pi}{k_m} = \frac{2\pi}{0.00029} = 21,545 \text{ ft}$$

Need to find speed of <u>red</u> <u>wave</u>

$$C_m = \frac{\omega_m}{k_m} = \frac{\frac{\omega_1-\omega_2}{2}}{\frac{k_1-k_2}{2}} = \frac{\omega_1-\omega_2}{k_1-k_2} = \frac{\Delta\omega}{\Delta k}$$

To the limit as $\Delta k \to 0 \Rightarrow \boxed{C_g = \text{group velocity} = \frac{d\omega}{dk}}$

Also $\boxed{C_g = C_{front}}$

Let's find c_g for deep water:

$$c_g = \frac{d\omega}{dk}$$

Dispersion relationship: $\frac{\omega^2}{k} = g$ (deep H_2O)

$$\Rightarrow \omega = \sqrt{gk}$$

$$\boxed{c_g = \frac{d\omega}{dk}} = \sqrt{g}\,\frac{d}{dk}k^{1/2} = \sqrt{g}\,\frac{1}{2}\,k^{(1-\frac{1}{2})} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

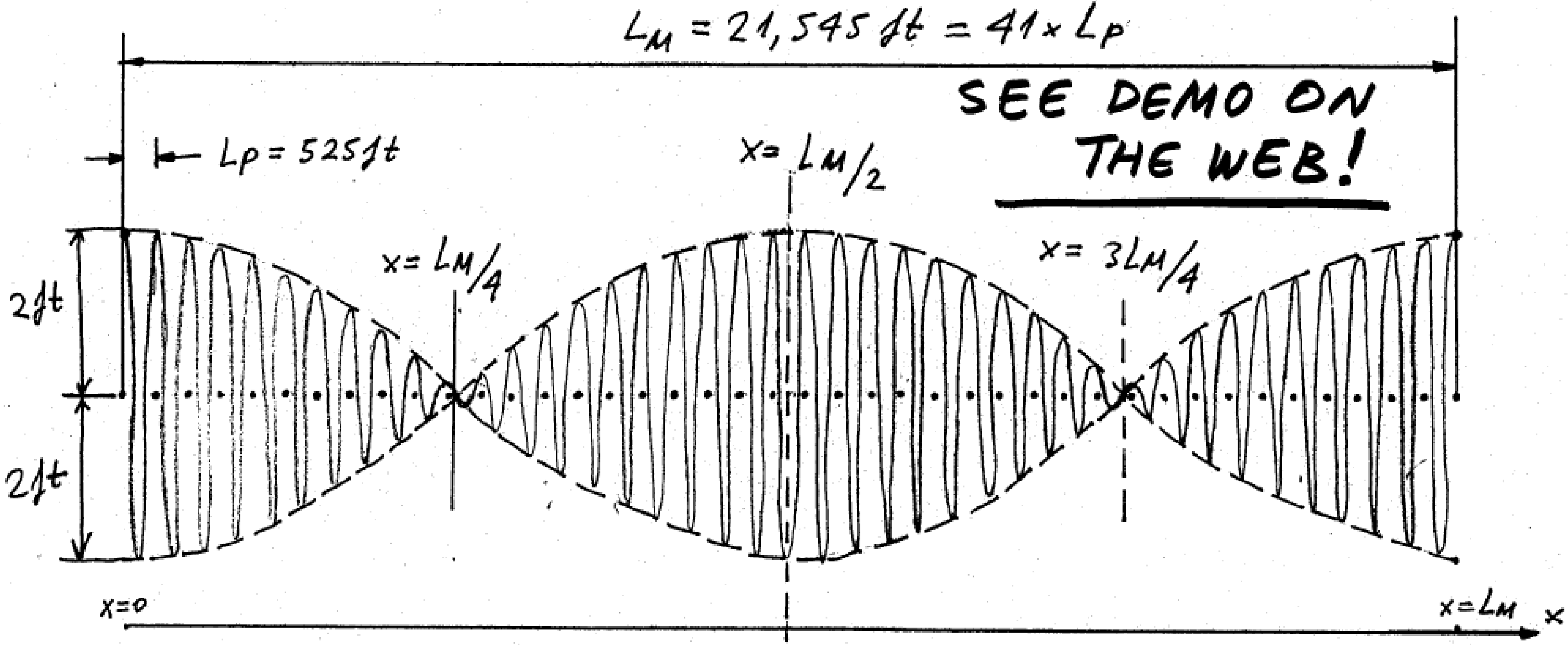
$$\boxed{C = \frac{\omega}{k}} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}}$$

$$\Rightarrow \boxed{c_g = \frac{c}{2}}$$

For deep H_2O

2 WAVES WITH SLIGHLTY DIFFERENT LENGTH/PERIOD

NOTE: THERE ARE 41 waves of wave length $\boxed{L_P}$ inside wave length $\boxed{L_M}$



SEE DEMO ON THE WEB!

Envelope: $\eta_e = H \cos\left[\frac{2\pi x}{L_M} - \frac{2\pi t}{T_M}\right]$ (propagates in the same direction as original)

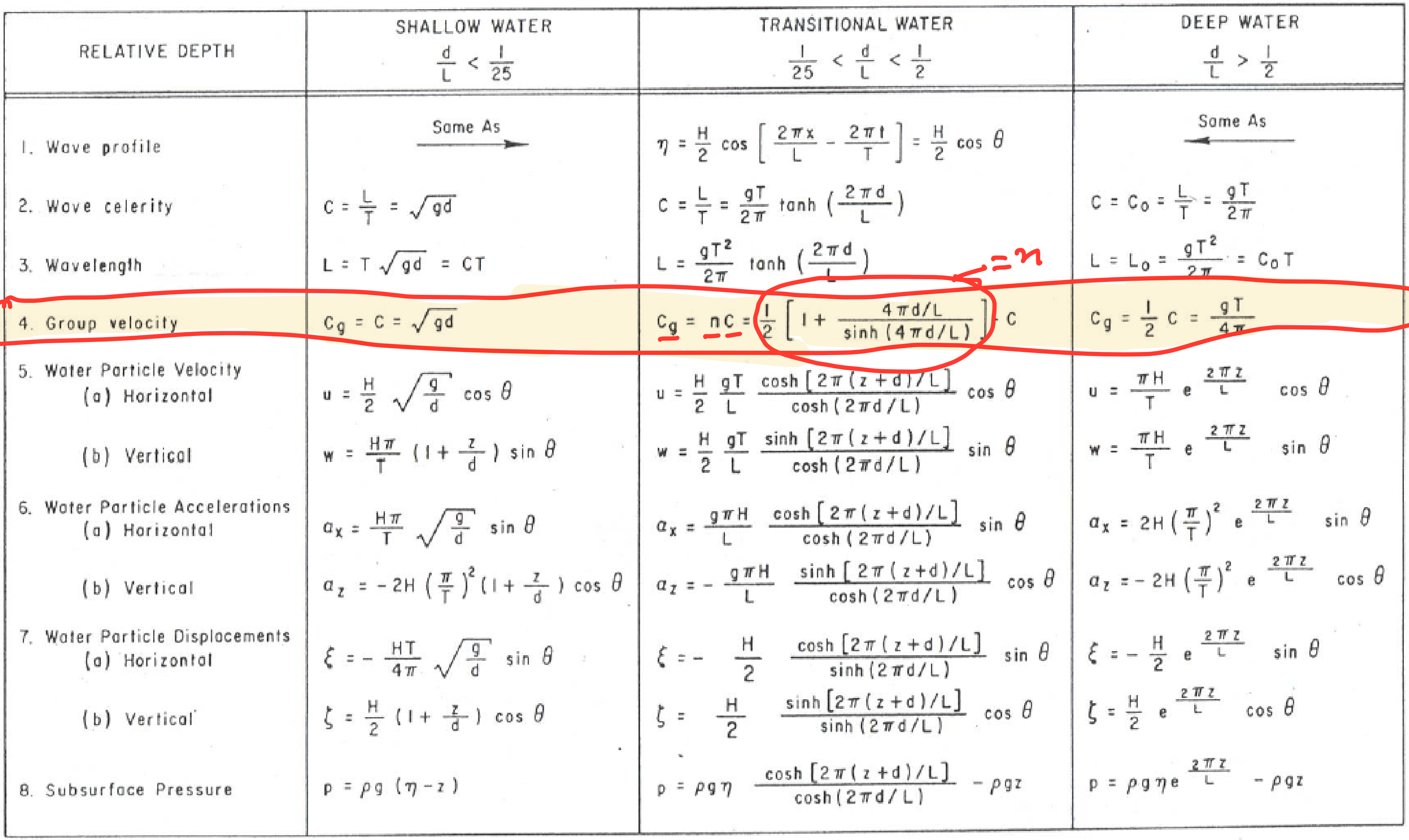
Wave speed of envelope is: $C_e = L_M/T_M = 21,545/820 = 26.27\ ft/sec$

group velocity of wave (1) (for deep water) $C_g = C_1/2 = 0.5\ L_1/T_1 = 25.62\ ft/sec$.

$C_e \approx C_g$ Actually definition of $C_g = d\omega/dk = \lim_{dk \to 0}(\overline{C_e})$.

since $\Delta k = k_1 - k_2 = 0.01226 - 0.01168 = 5.8 \times 10^{-4}$ not quite "0"

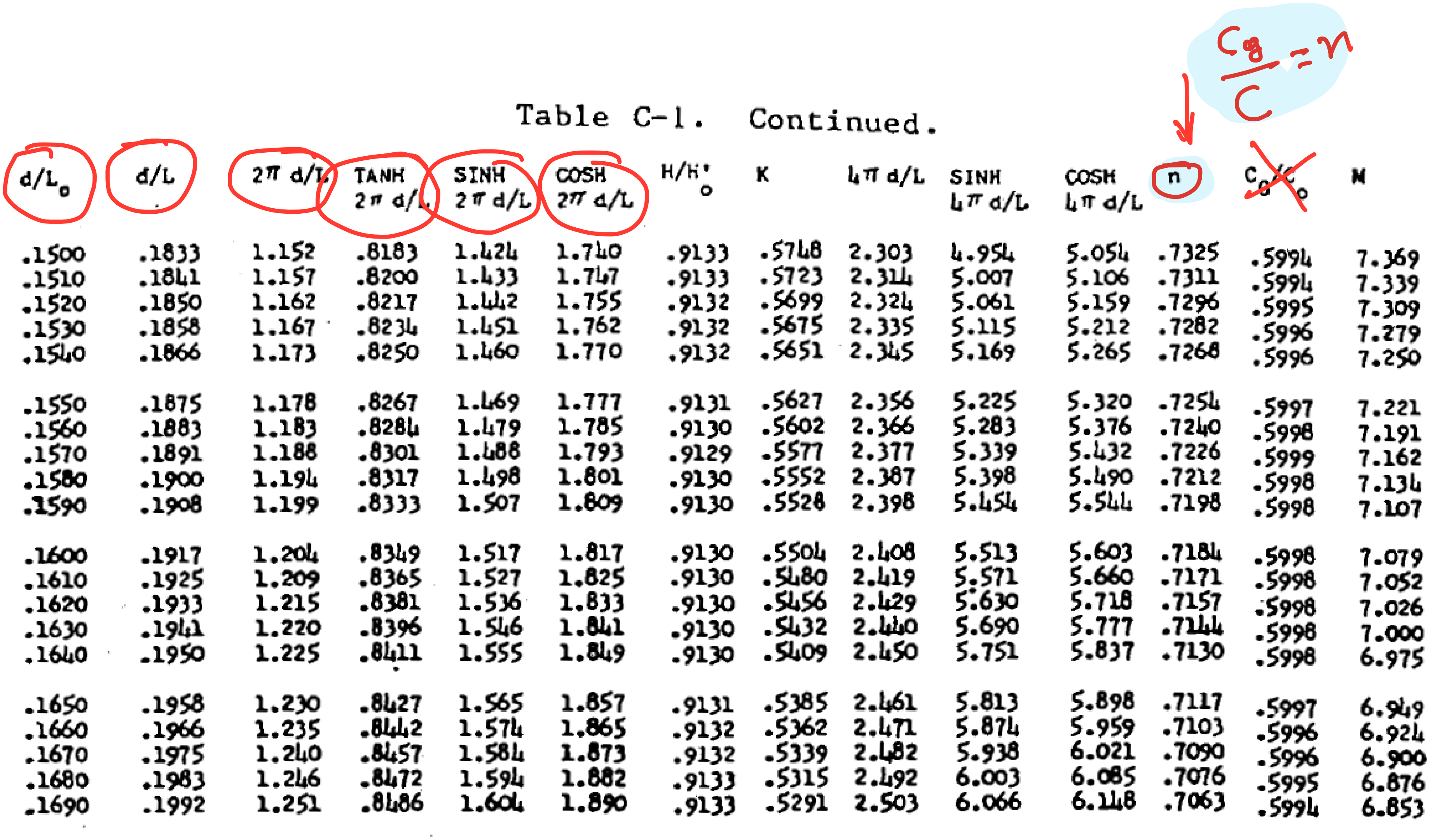
FORMULAS FOR GROUP VELOCITY



RELATIVE DEPTH	SHALLOW WATER $\frac{d}{L} < \frac{1}{25}$	TRANSITIONAL WATER $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	DEEP WATER $\frac{d}{L} > \frac{1}{2}$
1. Wave profile	Same As →	$\eta = \frac{H}{2}\cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	← Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_0 = \frac{gT^2}{2\pi} = C_0 T$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \overbrace{\frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right]}^{=n} C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water Particle Velocity (a) Horizontal	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$u = \frac{\pi H}{T}e^{\frac{2\pi z}{L}}\cos\theta$
(b) Vertical	$w = \frac{H\pi}{T}\left(1 + \frac{z}{d}\right)\sin\theta$	$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$w = \frac{\pi H}{T}e^{\frac{2\pi z}{L}}\sin\theta$
6. Water Particle Accelerations (a) Horizontal	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}}\sin\theta$
(b) Vertical	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1 + \frac{z}{d}\right)\cos\theta$	$a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}}\cos\theta$
7. Water Particle Displacements (a) Horizontal	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$	$\xi = -\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\sin\theta$	$\xi = -\frac{H}{2}e^{\frac{2\pi z}{L}}\sin\theta$
(b) Vertical	$\zeta = \frac{H}{2}\left(1 + \frac{z}{d}\right)\cos\theta$	$\zeta = \frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\cos\theta$	$\zeta = \frac{H}{2}e^{\frac{2\pi z}{L}}\cos\theta$
8. Subsurface Pressure	$p = \rho g(\eta - z)$	$p = \rho g\eta\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$	$p = \rho g\eta e^{\frac{2\pi z}{L}} - \rho g z$

Figure 2-6. Summary of linear (Airy) wave theory--wave characteristics.

HOW TO DETERMINE GROUP VELOCITY FROM TABLE C-1



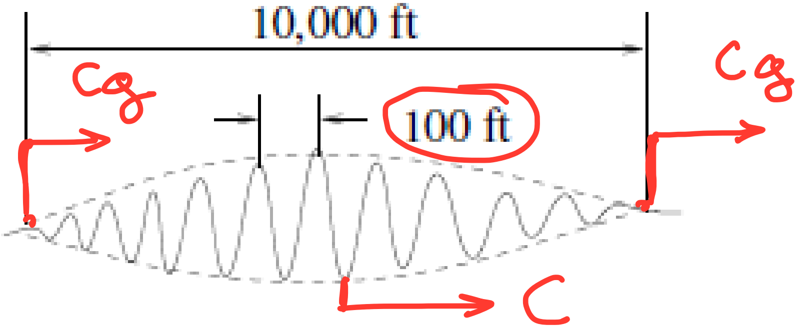
$\frac{C_g}{C} = n$

Table C-1. Continued.

d/L_o	d/L	$2\pi\ d/L$	TANH $2\pi\ d/L$	SINH $2\pi\ d/L$	COSH $2\pi\ d/L$	H/H'_o	K	$4\pi\ d/L$	SINH $4\pi\ d/L$	COSH $4\pi\ d/L$	n	~~C_G/C_o~~	M
.1500	.1833	1.152	.8183	1.424	1.740	.9133	.5748	2.303	4.954	5.054	.7325	.5994	7.369
.1510	.1841	1.157	.8200	1.433	1.747	.9133	.5723	2.314	5.007	5.106	.7311	.5994	7.339
.1520	.1850	1.162	.8217	1.442	1.755	.9132	.5699	2.324	5.061	5.159	.7296	.5995	7.309
.1530	.1858	1.167	.8234	1.451	1.762	.9132	.5675	2.335	5.115	5.212	.7282	.5996	7.279
.1540	.1866	1.173	.8250	1.460	1.770	.9132	.5651	2.345	5.169	5.265	.7268	.5996	7.250
.1550	.1875	1.178	.8267	1.469	1.777	.9131	.5627	2.356	5.225	5.320	.7254	.5997	7.221
.1560	.1883	1.183	.8284	1.479	1.785	.9130	.5602	2.366	5.283	5.376	.7240	.5998	7.191
.1570	.1891	1.188	.8301	1.488	1.793	.9129	.5577	2.377	5.339	5.432	.7226	.5999	7.162
.1580	.1900	1.194	.8317	1.498	1.801	.9130	.5552	2.387	5.398	5.490	.7212	.5998	7.134
.1590	.1908	1.199	.8333	1.507	1.809	.9130	.5528	2.398	5.454	5.544	.7198	.5998	7.107
.1600	.1917	1.204	.8349	1.517	1.817	.9130	.5504	2.408	5.513	5.603	.7184	.5998	7.079
.1610	.1925	1.209	.8365	1.527	1.825	.9130	.5480	2.419	5.571	5.660	.7171	.5998	7.052
.1620	.1933	1.215	.8381	1.536	1.833	.9130	.5456	2.429	5.630	5.718	.7157	.5998	7.026
.1630	.1941	1.220	.8396	1.546	1.841	.9130	.5432	2.440	5.690	5.777	.7144	.5998	7.000
.1640	.1950	1.225	.8411	1.555	1.849	.9130	.5409	2.450	5.751	5.837	.7130	.5998	6.975
.1650	.1958	1.230	.8427	1.565	1.857	.9131	.5385	2.461	5.813	5.898	.7117	.5997	6.949
.1660	.1966	1.235	.8442	1.574	1.865	.9132	.5362	2.471	5.874	5.959	.7103	.5996	6.924
.1670	.1975	1.240	.8457	1.584	1.873	.9132	.5339	2.482	5.938	6.021	.7090	.5996	6.900
.1680	.1983	1.246	.8472	1.594	1.882	.9133	.5315	2.492	6.003	6.085	.7076	.5995	6.876
.1690	.1992	1.251	.8486	1.604	1.890	.9133	.5291	2.503	6.066	6.148	.7063	.5994	6.853

EXAMPLE ON GROUP VELOCITY

A group of waves is 10,000 feet in length in deep water. The waves within the group are 100 feet in length.

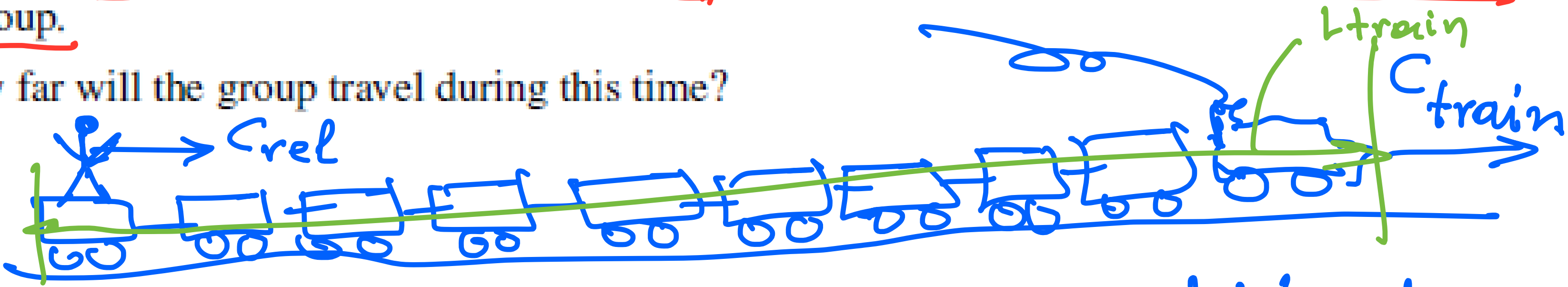


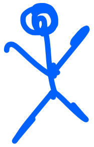
$$t_{cross} = \frac{10,000\ ft}{C_{rel}}$$

$$C_{rel} = C - C_g$$

a) Calculate the time required for a component wave to travel from the rear to the front of the group.

b) How far will the group travel during this time?





C = speed of runner w.r.t. outside observer

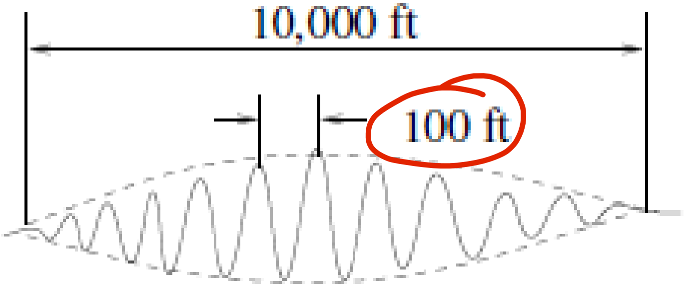
$$t_{cross} = \frac{L_{train}}{C_{rel}}$$

$$C_g \rightarrow C_{train}$$

$$C = C_{rel} + C_{train} \Rightarrow C_{rel} = C - C_{train}$$

EXAMPLE ON GROUP VELOCITY

A group of waves is 10,000 feet in length in deep water. The waves within the group are 100 feet in length.



$$C = \frac{L}{T} = \frac{100\,ft}{4.42\,s} = 22.64\,ft/s$$

$$L = \frac{gT^2}{2\pi} = 100\,ft = \frac{32.2 \times T^2}{2\pi} \Rightarrow$$

$$\Rightarrow T = 4.42\,sec$$

a) Calculate the time required for a component wave to travel from the rear to the front of the group.

b) How far will the group travel during this time?

In deep H_2O

$$C_g = \frac{C}{2} = 11.32\,ft/s$$

$$\Rightarrow C_{rel} = C - C_g = 22.64 - 11.32 = 11.32\,ft/s$$

a) $t_{cross} = \frac{10,000}{11.32} = 883.4\,sec$

b) $L_{group\ to\ travel} \longrightarrow C_g \cdot t_{cross} = 10,000\,ft$

GROUP OF 2 WAVES IN TRANSITIONAL WATER

Total wave profile and wave envelop for a group of two waves with a height of 1m each, and with wave lengths L_1=70 m (T_1=8.973 s) and L_2=73.5 m (T_2=9.37 s), at a depth of 7 m [L_{env}=$2L_1L_2/(L_2-L_1)$=2,940m]