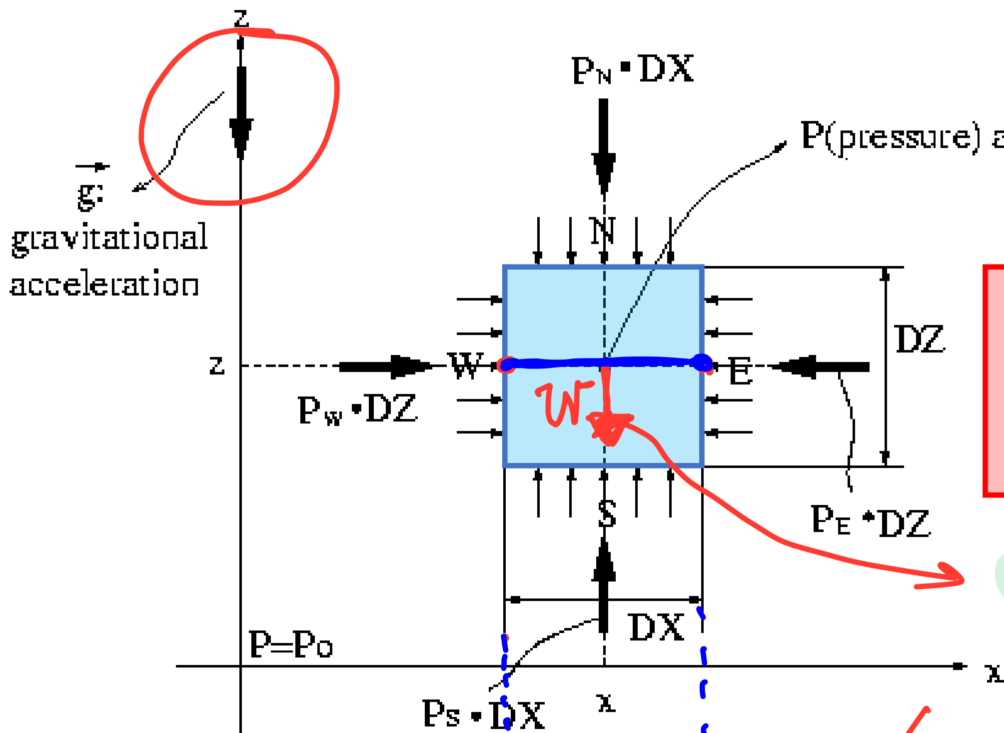
DEFINITION OF SUBSTANTIAL DERIVATIVE
(also called Substantive, Material, or Total derivative)

see previous set of slides

FORCES ON A PARTICLE



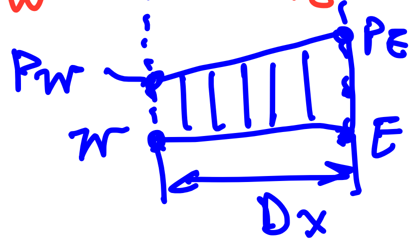
IGNORE VISCOUS (FRICTIONAL) FORCES

consistent with assumption of $\omega = 0$

$m = \rho \, Dx \, Dz$

$$\boxed{mg = \rho \, Dx \, Dz \, g}$$

$$\sum F_x = P_W \, DZ - P_E \, DZ = (P_W - P_E) \, DZ = -\frac{\partial P}{\partial x} Dx \, Dz$$



$$\frac{P_E - P_W}{Dx} = \frac{\partial P}{\partial x} \quad \text{(as } Dx \to 0\text{)}$$

$$(P_E - P_W) = \frac{\partial P}{\partial x} DX$$

or

$$P_W - P_E = -\frac{\partial P}{\partial x} Dx$$

$$\Rightarrow \sum F_x = -\frac{\partial P}{\partial x} Dx \, Dz$$

EULER EQUATIONS (=MOMENTUM EQUATIONS FOR INVISCID FLOW)

$\Sigma F_x = m a_x$

$-\frac{\partial p}{\partial x} \cancel{Dx}\,\cancel{Dz} = \rho\, \cancel{Dx}\,\cancel{Dz}\, a_x$

$\boxed{-\frac{\partial p}{\partial x} = \rho a_x}$

Euler equation along x

$\Sigma F_z = m a_z$

$\boxed{-\frac{\partial p}{\partial z} - \rho g = \rho a_z}$

Euler equ. along z

$$\rho\underbrace{\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right]}_{=a_x} \equiv \rho\underbrace{\frac{Du}{Dt}}_{=a_x} = -\frac{\partial p}{\partial x} \qquad \text{Momentum along x} \qquad (80)$$

$$\rho\underbrace{\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right]}_{=a_z} \equiv \rho\underbrace{\frac{Dw}{Dt}}_{=a_z} = -\rho g - \frac{\partial p}{\partial z} \qquad \text{Momentum along z} \qquad (81)$$

GENERALIZED BERNOULLI EQUATION

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right]\equiv\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}$$ Euler Momentum along x (80)

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right]\equiv\rho\frac{Dw}{Dt}=-\rho g-\frac{\partial p}{\partial z}$$ Euler Momentum along z (81)

$\omega=0$ or $\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0 \sim \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}$

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial w}{\partial x}\right]=-\frac{\partial p}{\partial x}$$

$$\frac{1}{2}\frac{\partial u^2}{\partial x}=\frac{1}{2}\frac{\partial u^2}{\partial u}\frac{\partial u}{\partial x}=\frac{1}{\cancel{2}}\cancel{2}u\frac{\partial u}{\partial x}$$

$2u$

$$\frac{1}{2}\frac{\partial w^2}{\partial x}$$

$$\frac{1}{2}\frac{\partial u^2}{\partial x}+\frac{1}{2}\frac{\partial w^2}{\partial x}=\frac{\partial}{\partial x}\left[\frac{u^2+w^2}{2}\right]=\frac{\partial}{\partial x}\left[\frac{q^2}{2}\right]$$

GENERALIZED BERNOULLI EQUATION

$$\rho\left[\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q^2}{2}\right)\right]=-\frac{\partial p}{\partial x}$$

$$u=\frac{\partial \phi}{\partial x} \qquad \frac{\partial}{\partial t}\frac{\partial \phi}{\partial x}=\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t}$$

$$\rho\left[\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}\left(\frac{q^2}{2}\right)\right]+\frac{\partial p}{\partial x}=0$$

$$\frac{\partial}{\partial x}\left[\underbrace{\rho\frac{\partial \phi}{\partial t}+\rho\frac{q^2}{2}+p+\rho g z}_{=H}\right]=0$$

Similarly starting from Euler equ. along z:

$$\frac{\partial}{\partial z}\left[\underbrace{\rho\frac{\partial \phi}{\partial t}+\rho\frac{q^2}{2}+p+\rho g z}_{H}\right]=0$$

GENERALIZED BERNOULLI EQUATION

$$H = \rho \frac{\partial \phi}{\partial t} + \rho \frac{q^2}{2} + p + \rho g z$$

$$\boxed{\begin{aligned} \frac{\partial H}{\partial x} &= 0 \\ \frac{\partial H}{\partial z} &= 0 \end{aligned}}$$

Should apply EVERYWHERE in the flowfield

$$\Rightarrow \underline{\underline{H = \text{constant}!!}}$$

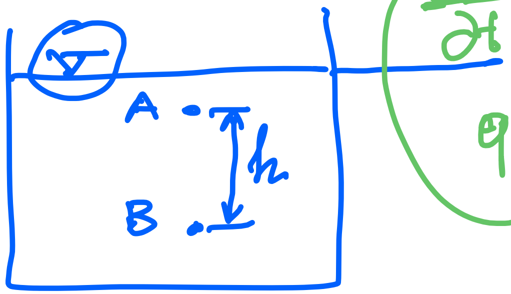
$$\boxed{H = \rho \frac{\partial \phi}{\partial t} + \rho \frac{q^2}{2} + p + \rho g z = \text{Constant}}$$

Generalized Bernoulli equ.

- Applies to <u>steady</u> & <u>unsteady</u> flow
- Applies among any points in the flowfield and <u>not</u> only along the same streamline

SPECIAL CASES OF BERNOULLI EQUATION

Standing fluid



$\frac{\partial \phi}{\partial t} = 0$

$q = 0$

$p + \rho g z = \text{Constant}$

$p_A + \rho g z_A = p_B + \rho g z_B$

$\boxed{p_B} = p_A + \rho g (z_A - z_B) =$

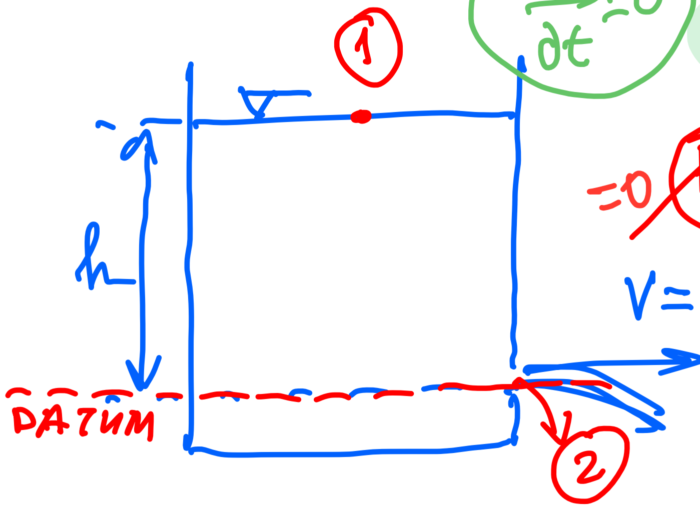
$= \boxed{p_A + \rho g h}$

hydrostatic law

if A on the free surface $p_A(\text{gage}) = 0$

$$\boxed{p_B = \rho g h}$$

SPECIAL CASES OF BERNOULLI EQUATION



$$\frac{\partial\phi}{\partial t}=0$$

$$p+\rho\frac{q^2}{2}+\rho g z=\text{Constant}$$

$$\underbrace{\cancel{p_1}}_{=0}+\rho\cancel{\frac{q_1^2}{2}}+\rho g \underbrace{z_1}_{h}=\underbrace{p_2}_{=0}+\rho\underbrace{\frac{q_2^2}{2}}_{\frac{V^2}{2}}+\rho g \underbrace{z_2}_{0}$$

$q_1 \approx 0$

$$\boxed{V=\sqrt{2gh}}$$

Summary of important equations (so far...)

Equations for unsteady irrotational/incompressible fluid flow in 2-D:

a) $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$ (continuity or conservation of mass equation)

b) $\frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$ (irrotationality condition when vorticity $\omega = 0$)

Momentum or Euler equations

c) $\rho a_x = -\frac{\partial p}{\partial x}$ (c1)

(with acceleration: $a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}$)

$\rho a_z = -\rho g - \frac{\partial p}{\partial z}$ (c2)

(with acceleration: $a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}$)

With the introduction of velocity potential Φ:

The velocities u, w are given as:

✓ $u = \frac{\partial \Phi}{\partial x}$, $w = \frac{\partial \Phi}{\partial z}$ ✓

→ Equ. (a) above becomes: $\boxed{\nabla^2 \Phi \equiv \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0}$ (Laplace Equ.) (a')

→ Equ. (b) is automatically satisfied

→ Equs (c) become the Bernoulli equ:

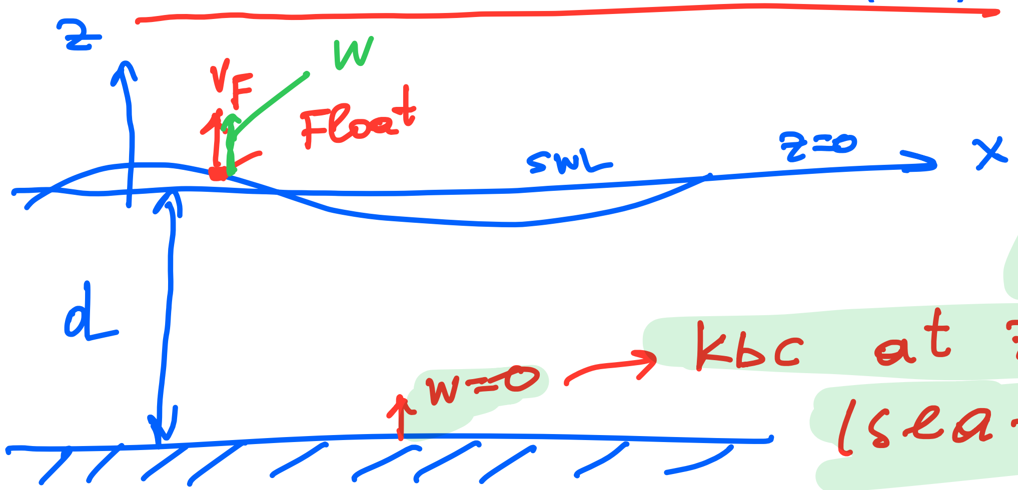
$$\rho \frac{\partial \Phi}{\partial t} + p + \frac{\rho q^2}{2} + \rho g z = \text{constant} \quad (c')$$

where: $q = \sqrt{u^2 + w^2}$

* (a') + (c') apply at ALL POINTS IN THE FLOW FIELD

✓ Need boundary conditions to determine unique Φ for a given physical problem

KINEMATIC BOUNDARY CONDITION (KBC)



kbc at $z=-d$ (sea-floor)

Physical meaning: particles cannot move normal to solid boundaries

$w = V_F$

$$\boxed{\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}}$$

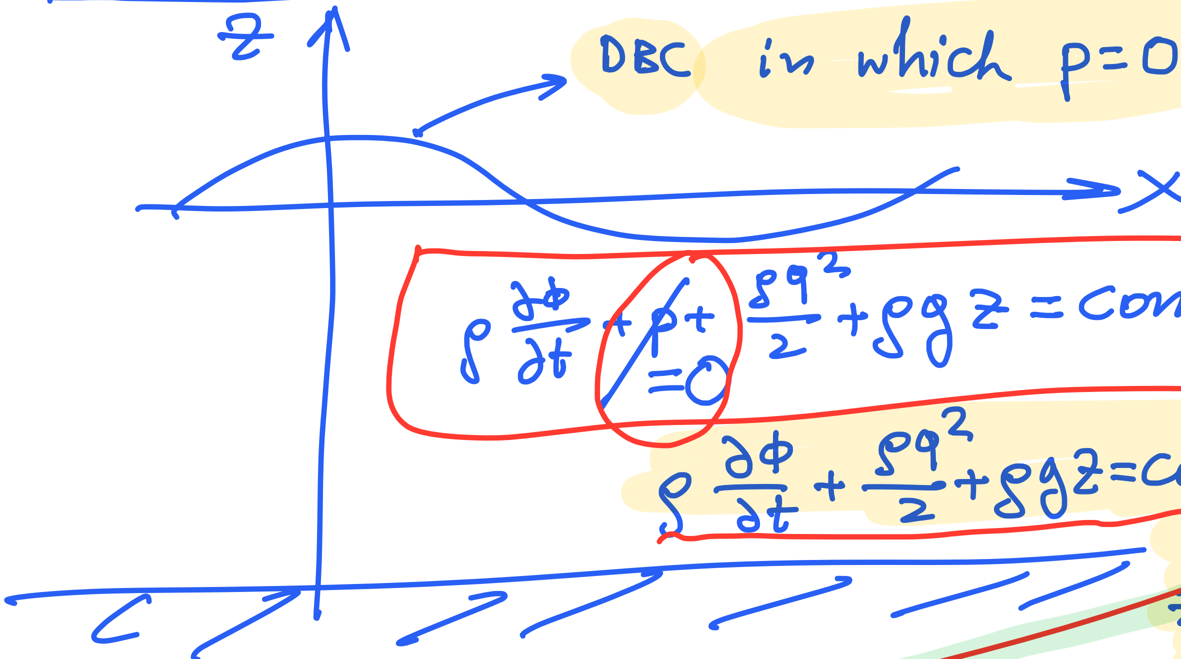
kbc on the free-surface thus applies at $z=\eta$

Physical meaning:

Float and fluid particle next to it have to move together along the vertical

KINEMATIC BOUNDARY CONDITION (KBC) ON THE FREE SURFACE

DYNAMIC BOUNDARY CONDITION (DBC) ON THE FREE SURFACE



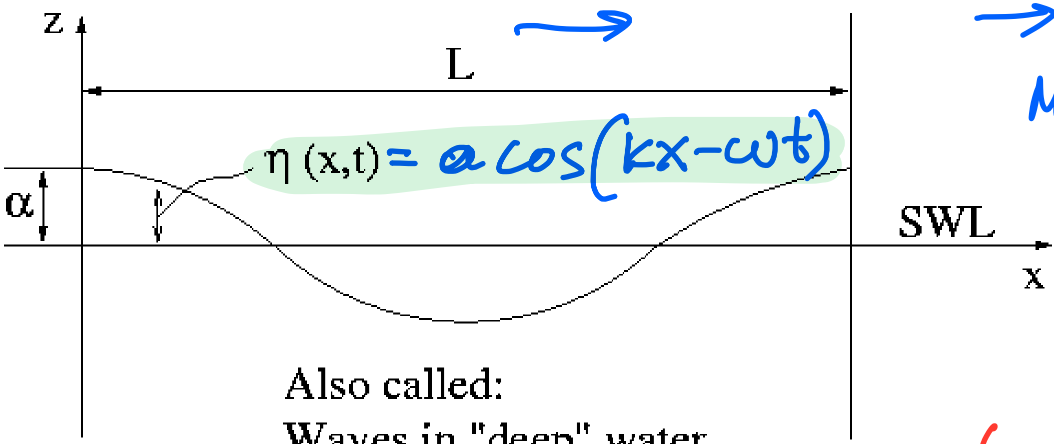
DBC in which $p=0$ at $z=\eta$

$$\boxed{\rho \frac{\partial \phi}{\partial t} + \underbrace{\cancel{p}}_{=0} + \frac{\rho q^2}{2} + \rho g z = const.}$$

$$\rho \frac{\partial \phi}{\partial t} + \frac{\rho q^2}{2} + \rho g z = const. \quad \text{at } z=\eta$$

Physical meaning:
pressure on the free-surface must be equal to atmospheric at all locations and at all times (or $p_{gage}=0$)

LINEAR WAVE THEORY – "INFINITE" DEPTH WATER or DEEP WATER



$\rightarrow \phi(x,z,t) = A e^{\lambda z} \sin(kx-\omega t)$

Must determine λ, A

From $\nabla^2\phi = 0 \Rightarrow$

$\lambda = \pm K$

$\lambda = k$ acceptable solution

(see previous set of slides)

Kbc on the free surface:

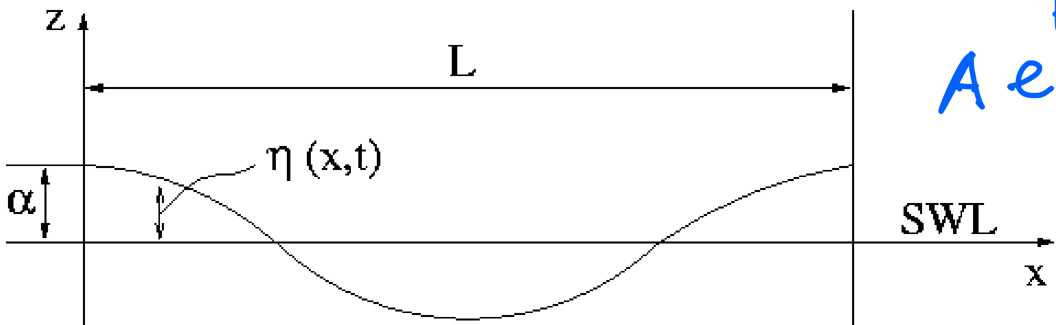
$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \text{ at } z=\eta$$

$$\frac{\partial \phi}{\partial z} = A e^{\lambda z} \lambda \sin(kx-\omega t)$$

$$\frac{\partial \eta}{\partial t} = a\left[-\sin(kx-\omega t)\right](-\omega) = a\omega \sin(kx-\omega t)$$

$$\|$$

LINEAR WAVE THEORY – DEEP WATER



Also called:
Waves in "deep" water

$(\cancel{\lambda = k})$

$$A e^{kz} \cdot k \sin(\cancel{kx-\omega t}) = a\omega \sin(\cancel{kx-\omega t})$$

(at $z=\eta$)

$$A e^{k\eta} \cdot k = a\omega$$

$$\Rightarrow A = \frac{a\omega}{k} e^{-k\eta}$$

Assumption of Linear Theory

$a << L$ or $\frac{a}{L} << 1$

Linear wave theory

$$k\eta = \frac{2\pi}{L} a \cos(kx-\omega t) << 1$$

$$e^{-k\eta} \approx e^{-0} = 1$$

$$\Rightarrow \boxed{A = \frac{a\omega}{k}}$$

or Airy theory

(or Small Amplitude Wave Theory (SAWT)

$\lambda = k$, $A = \frac{a\omega}{k}$ $\quad \phi(x,z,t) = \frac{a\omega}{k} e^{kz} \sin(kx - \omega t)$

attenuation factor

DBC on the free surface: $(p = 0)$ at $z = \eta$

$$\cancel{\rho}\frac{\partial \phi}{\partial t} + \cancel{\rho \frac{q^2}{2}} + \cancel{\rho} g z = 0$$

$z = a\cos(kx - \omega t) = \eta$

$q^2 = u^2 + w^2$

$u = \frac{\partial \phi}{\partial x} = \frac{a\omega}{\cancel{k}} e^{kz} \cos(kx - \omega t)\cancel{k}$

$w = \frac{\partial \phi}{\partial z} = \frac{a\omega}{\cancel{k}} e^{kz} \cancel{k} \sin(kx - \omega t)$

$$= a^2\omega^2 e^{2kz} \underbrace{\left[\cos^2(kx - \omega t) + \sin^2(kx - \omega t)\right]}_{=1}$$

$\Rightarrow q^2 = a^2\omega^2 e^{2kz}$

or $\boxed{q = a\omega e^{kz}}$

$$\boxed{\frac{\omega^2}{k} = g}$$

$$\frac{\partial \phi}{\partial t} = \frac{a\omega}{k} e^{kz} \left(\cos(kx - \omega t)\right)(-\omega) = -\frac{a\omega^2}{k} e^{kz} \cos(kx - \omega t)$$

$$\frac{\partial \phi}{\partial t} + \frac{q^2}{2} + gz = 0$$

$$-\frac{\cancel{a}\,\omega^2}{k} e^{kz} \cancel{\cos(kx-\omega t)} + \cancel{\frac{a^2 \omega^2 e^{2kz}}{2}} + g\,\cancel{a \cos(kx-\omega t)}$$

H.O.T. $\quad = 0$

$$-\frac{\omega^2}{k} e^{k\eta} + g = 0 \qquad e^{k\eta} \approx 0 \quad \Longrightarrow \quad \boxed{\frac{\omega^2}{k} = g}$$

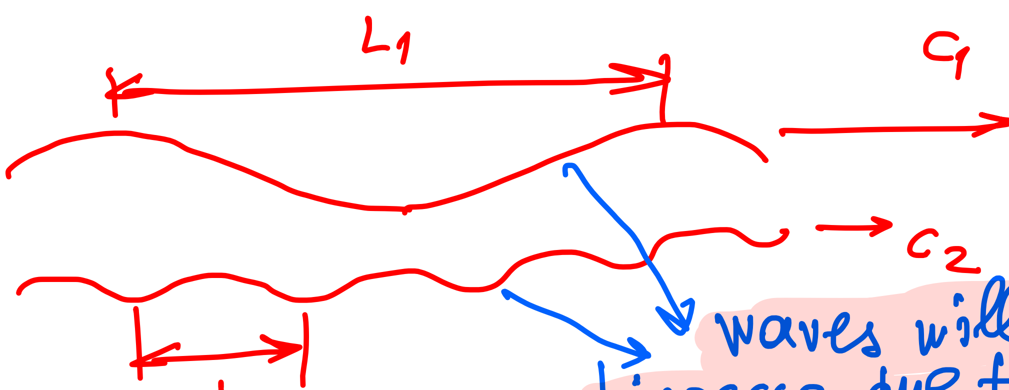
dispersion relationship for deep water

$$\omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{L}$$

$$\frac{\left(\frac{2\pi}{T}\right)^2}{(2\pi/L)} = g \rightarrow \boxed{L = \frac{gT^2}{2\pi}}$$

$$\boxed{c = \frac{L}{T}} = \frac{\frac{gT^2}{2\pi}}{T} = \boxed{\frac{gT}{2\pi}}$$

$$\boxed{T = \sqrt{\frac{2\pi L}{g}}} \rightarrow \boxed{c = \sqrt{\frac{gL}{2\pi}}}$$



waves will disperse due to different speeds!

$L_1 > L_2 \rightsquigarrow c_1 > c_2$

Thus name "dispersion" relationship

From c, L, T <u>ONLY</u> one can be specified the others are expressed in terms of it!

$$\phi = \frac{a\omega}{k} e^{kz} \sin(kx-\omega t)$$

$$\boxed{u = \frac{\partial \phi}{\partial x} = a\omega e^{kz} \cos(kx-\omega t)}$$

$$\boxed{w = \frac{\partial \phi}{\partial z} = a\omega e^{kz} \sin(kx-\omega t)}$$

wave is not felt under this level $\Rightarrow$

$$kz = \frac{\cancel{2}\pi}{\cancel{L}}\left(-\frac{\cancel{L}}{\cancel{2}}\right) = -\pi$$

$$\underline{e^{kz}} = e^{-\pi} = \underline{0.0432}$$

Practical limit for "deep" water:

$\boxed{d > \frac{L}{2}}$ or $\boxed{\frac{d}{L} > \frac{1}{2}}$

Accelerations in deep H_2O

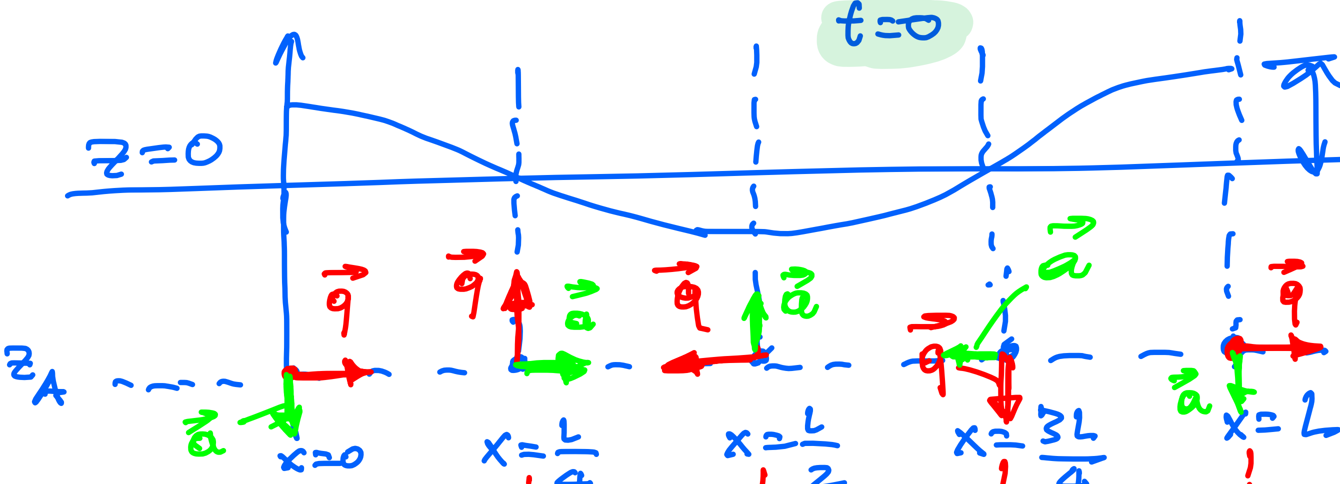
$$a_x = \underbrace{\frac{\partial u}{\partial t}}_{\sim a} + \cancel{\underbrace{u}_{\sim a}\,\underbrace{\frac{\partial u}{\partial x}}_{\sim a}} + \cancel{\underbrace{w}_{\sim a}\,\underbrace{\frac{\partial u}{\partial z}}_{\sim a}}$$

H.O.T. !!

$$a_z = \underbrace{\frac{\partial w}{\partial t}}_{\sim a} + \cancel{\underbrace{u}_{\sim a}\,\underbrace{\frac{\partial w}{\partial x}}_{\sim a}} + \cancel{\underbrace{w}_{\sim a}\,\underbrace{\frac{\partial w}{\partial z}}_{\sim a}}$$

$$\boxed{a_x = \frac{\partial u}{\partial t} = a\omega^2 e^{kz} \sin(kx - \omega t)}$$

$$\boxed{a_z = \frac{\partial w}{\partial t} = -a\omega^2 e^{kz} \cos(kx - \omega t)}$$



$u = a\omega e^{kz}\cos(kx - \cancel{\omega t})$

$W = a\omega e^{kz}\sin(kx - \cancel{\omega t})$

e^{kz_A} is fixed

$kx = \frac{2\pi}{L}x$

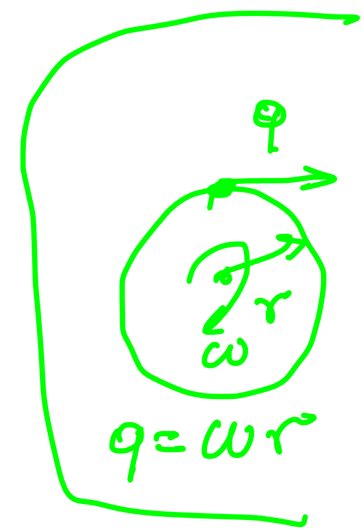
kx	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

$x=0$: $u = a\omega e^{kz_A}$, $W = 0$

$x=\frac{L}{4}$: $u = a\omega e^{kz_A}\cdot 0 = 0$, $W = a\omega e^{kz_A}$

$x=\frac{L}{2}$: $u = -a\omega e^{kz_A}$, $W = 0$

$x=\frac{3L}{4}$: $u = 0$, $W = -a\omega e^{kz_a}$



Trajectory of particle should be circular:

$q = a\cancel{\omega}e^{kz}$

$q = \cancel{\omega}r$

$\boxed{r = ae^{kz}}$

$a_x = a\omega^2 e^{kz}\sin(kx - \cancel{\omega t})$

$a_z = -a\omega^2 e^{kz}\cos(kx - \cancel{\omega t})$

$x = 0$: $a_x = 0$, $a_z = -a\omega^2 e^{kz}$

$x = \frac{L}{4}$: $a_x = a\omega^2 e^{kz}$, $a_z = 0$

For $x = L/2$ & $3L/4$ $\vec{a}$ is shown on the graph above.

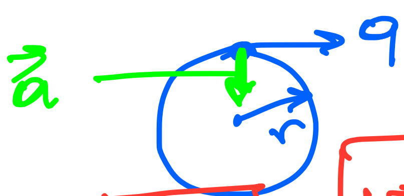
$$|\vec{a}| = \sqrt{a_x^2 + a_z^2} =$$

$$= \sqrt{a^2\omega^4 e^{2kz}\sin^2(kx-\omega t) + a^2\omega^4 e^{2kz}\cos^2(kx-\omega t)} \Rightarrow$$

$$|\vec{a}| = \boxed{a\omega^2 e^{kz}}$$

SAME!!

Remember $\boxed{q = a\omega e^{kz}}$



$\boxed{q = \omega r}$

$r = a e^{kz}$

$$\boxed{|\vec{a}| = \frac{q^2}{r} = \omega^2 r = q\omega}$$

check with formula of object in circular path

$$|\vec{a}| = \frac{q^2}{r} = \frac{a^2\omega^2 e^{2kz}}{a\, e^{kz}}$$

$$= a\omega^2 e^{kz}$$