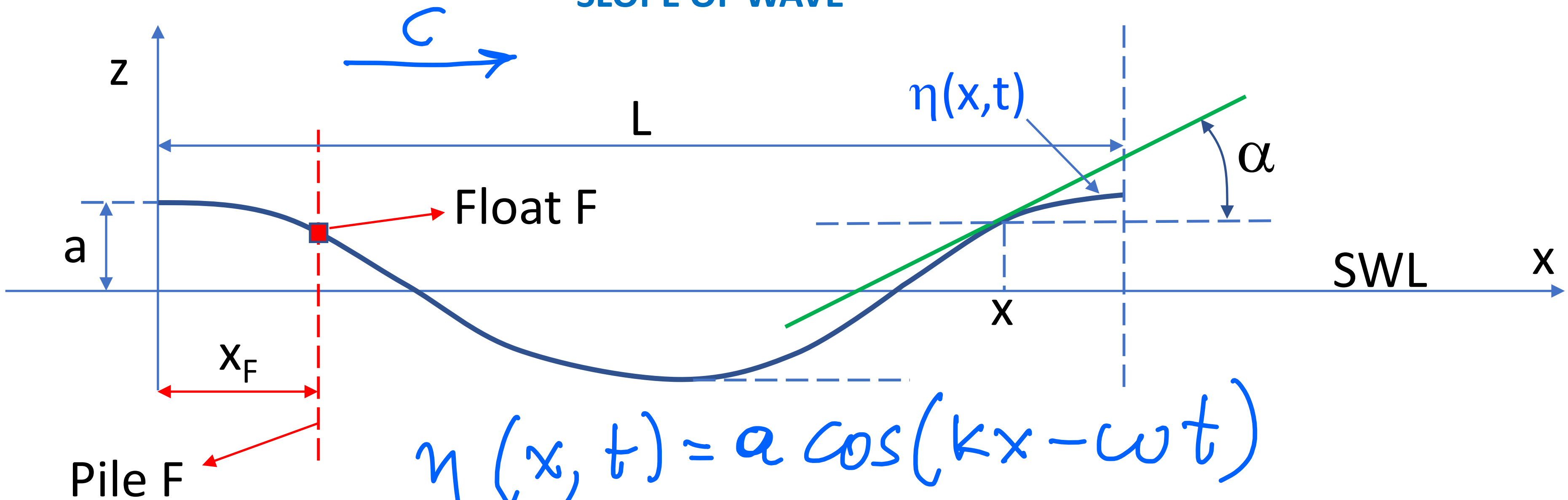
SLOPE OF WAVE



$$\eta(x,t) = a\cos(kx-\omega t)$$

Slope of wave $= \tan(\alpha) = \dfrac{\partial \eta}{\partial x}$; α = slope angle

$$\frac{\partial \eta}{\partial x} = a\left[-\sin(kx-\omega t)\right]k = -ak\sin(kx-\omega t)$$

max slope is $ak = 2\pi \dfrac{a}{L}$

remember:

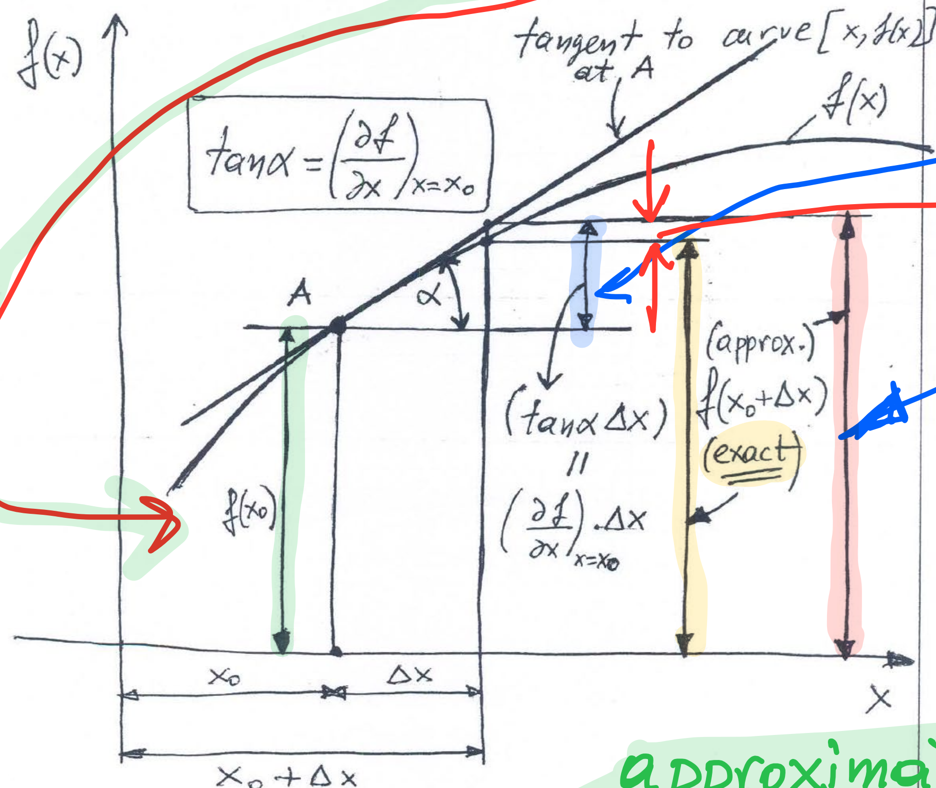
$$\frac{\partial \cos x}{\partial x} = -\sin x$$

$$\frac{\partial \sin x}{\partial x} = \cos x$$

In general if $f = A\cos(kx \pm \omega t)$ or $f = A\sin(kx \pm \omega t)$ then $\max f = |A|$ & $\min f = -|A|$

PHYSICAL MEANING OF 1st ORDER APPROXIMATION

$$f(x_0+\Delta x) \approx f(x_0) + \left(\frac{\partial f}{\partial x}\right)_{x=x_0} \cdot \Delta x = f(x_0) + (\tan\alpha)\,\Delta x$$



HOT = exact $f(x_0+\Delta x)$ − appr. $f(x_0+\Delta x)$

approximate term (Linear approx) (or 1st order approx)

Taylor expansion:

$$f(x_0+\Delta x) = f(x_0) + f'(x_0)\frac{\Delta x}{1!} + f''(x_0)\frac{(\Delta x)^2}{2!} + f'''(x_0)\cdot\frac{(\Delta x)^3}{3!} + \cdots$$

Higher Order Terms } HOT are negligible for small Δx

EXAMPLE ON 1st ORDER APPROXIMATION

Provide an *approximation* (first order expansion) of the following expression, H, for *small* values of a. Then apply the formula for $a = 0.1$ and compare the answer to the exact value. Determine the percentage error: $(H_{approx} - H_{exact})/H_{exact} = ?$

$$H = \sqrt{1+\alpha} = ? \tag{1}$$

Linear Taylor exp: $f(x_0+\Delta x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} \Delta x$

of our problem: $H \to f(x) = \sqrt{x}$, $x_0 \to 1$, $\Delta x \to \alpha$

$$\frac{df}{dx} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{1-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \Rightarrow \left.\frac{df}{dx}\right|_{x_0=1} = \frac{1}{2}$$

remember!
$$\frac{d}{dx} x^n = n x^{n-1}$$

Thus:
$$\boxed{H \approx 1 + \frac{1}{2} a = 1 + \frac{a}{2}}$$

EXAMPLE ON 1st ORDER APPROXIMATION

Provide an *approximation* (first order expansion) of the following expression, H, for *small* values of a. Then apply the formula for $a = 0.1$ and compare the answer to the exact value. Determine the percentage error: $(H_{approx} - H_{exact})/H_{exact} = ?$

$$H = \sqrt{1+a} = ? = 1 + \frac{a}{2} \tag{1}$$

Try $a = 0.1$:

$$H_{exact} = \sqrt{1+0.1} = \sqrt{1.1} = 1.0488$$

$$H_{approx} = 1 + \frac{0.1}{2} = 1.05$$

$$\%\ error = \frac{1.05 - 1.0488}{1.0488} = 0.00114 = 0.11\%$$

↳ very small!

Generalized binomial approx: $(1+\alpha)^r \simeq 1 + r\alpha$

r = real number

α small

$(|\alpha| < 1)$

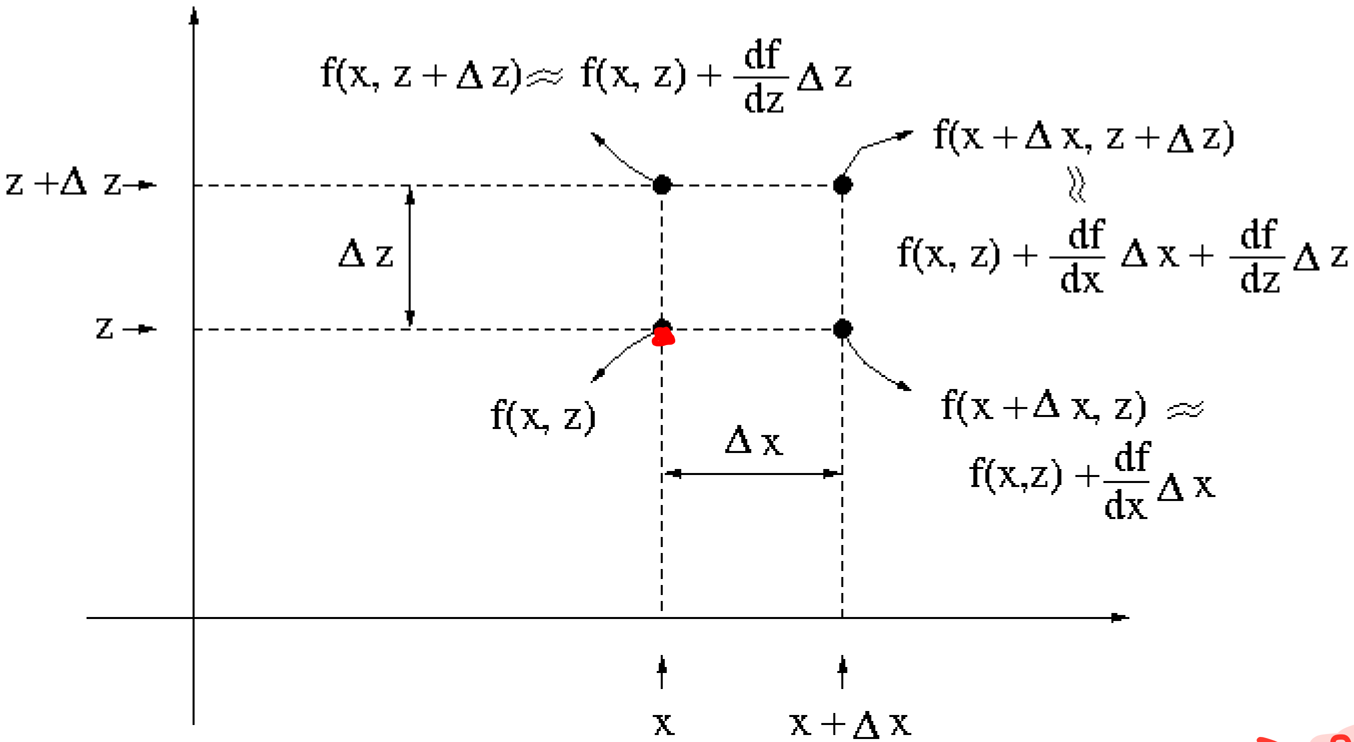
e.g. $\frac{1}{1+\alpha} = (1+\alpha)^{-1} \simeq 1 - \alpha$

Other useful approximations that can be derived using linear Taylor expansion:

$$e^x \approx 1 + x \quad \text{(small } x\text{)}$$

$$\tan\theta \approx \theta \quad \text{(small } \theta \text{ in \underline{RADIANS!}})$$

1st ORDER APPROXIMATION IN 2-D



$$f(x+\Delta x, z+\Delta z) \approx f(x,z) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial z}\Delta z + \text{HOT} \quad (19)$$

negligible for small $\Delta x, \Delta z$

these partial derivatives must be evaluated at (x, z)

EXAMPLE ON 1st ORDER APPROXIMATION IN 2-D

$$f(x,z) = x^3 + xz^2$$

$x=1,\ z=1$

$\Delta x = 0.1,\ \Delta z = 0.1$

$$f(1.1, 1.1) = f(1,1) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial z}\Delta z$$

$f(1,1) = 2$, $\Delta x = 0.1$, $\Delta z = 0.1$

$$\frac{\partial f}{\partial x} = 3x^2 + z^2 \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 4$$

$$\frac{\partial f}{\partial z} = 2zx \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{(1,1)} = 2$$

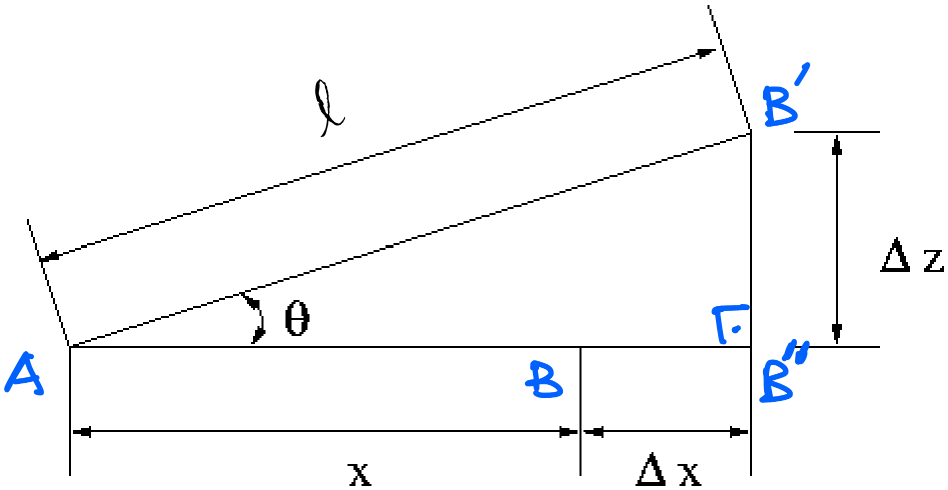
$\rightarrow 2 + 4 \times 0.1 + 2 \times 0.1 = 2.6$ (approx)

$f(1.1, 1.1) = 2.662$ exact

-2.3% error

SOME USEFUL APPROXIMATIONS

Q: How can ℓ and θ be approximated for ``small'' values of $\Delta x, \Delta z$ compared to x?



$AB' \approx AB'' = x + \Delta x$

$\theta = \tan\theta = \frac{\Delta z}{x+\Delta x} =$ (in radians!)

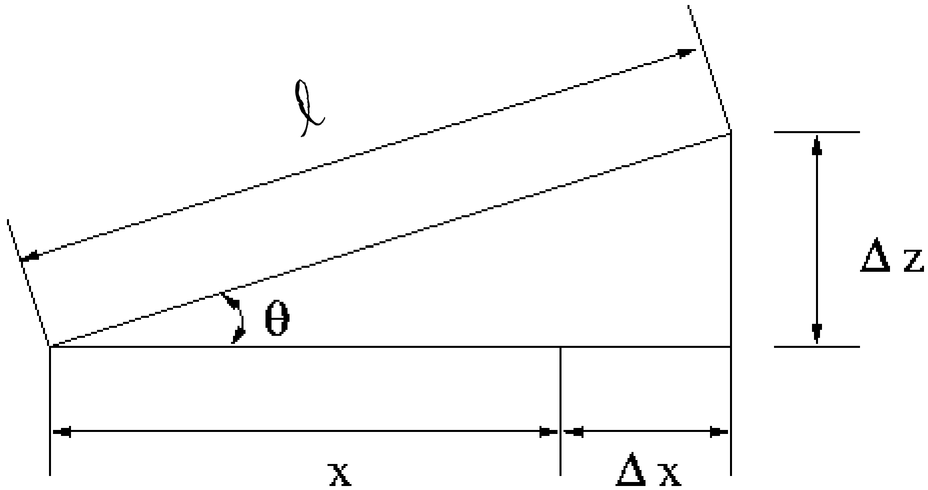
$= \frac{\Delta z}{x\left[1+\frac{\Delta x}{x}\right]} = \frac{\Delta z}{x}\left[1 - \frac{\Delta x}{x}\right] = \frac{\Delta z}{x} - \underbrace{\left(\overset{\ll 1}{\frac{\Delta z}{x}}\right)\left(\overset{\ll 1}{\frac{\Delta x}{x}}\right)}_{HOT} = \frac{\Delta z}{x}$

we used earlier approximation $\frac{1}{1+a} = 1-a$

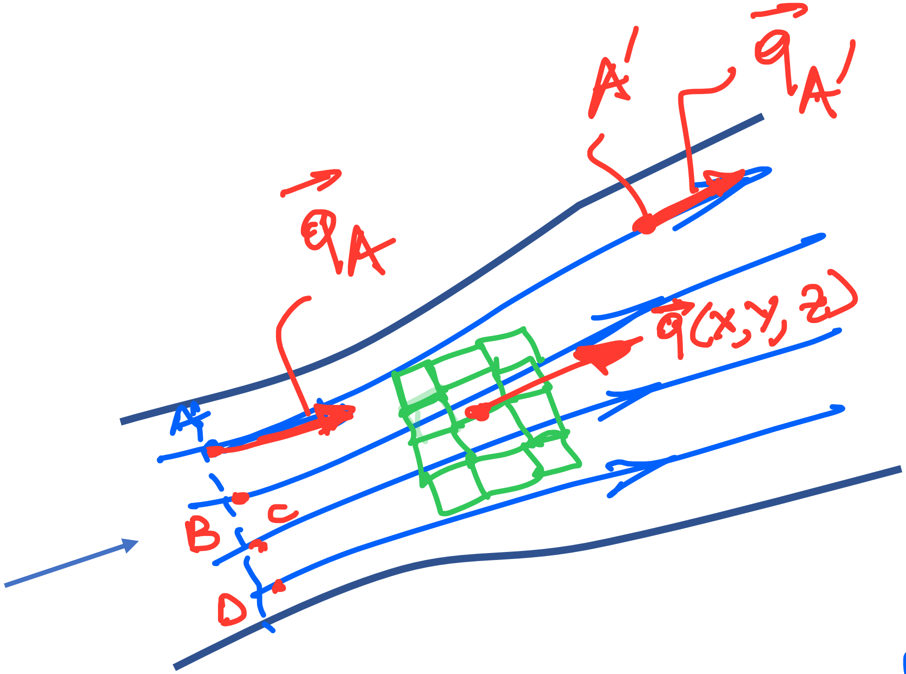
where $a = \frac{\Delta x}{x}$ ~ small since $\Delta x \ll x$

SOME USEFUL APPROXIMATIONS

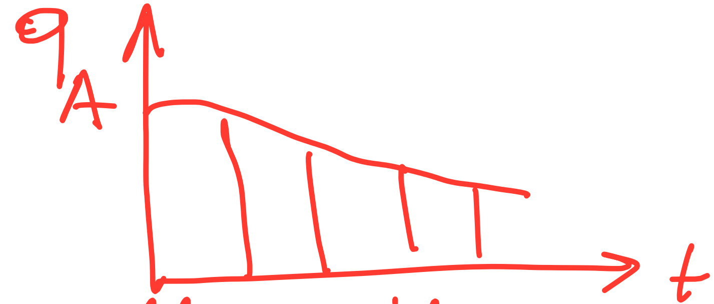
Q: How can ℓ and θ be approximated for ``small'' values of $\Delta x, \Delta z$ compared to x?



TWO APPROACHES TO DEFINE A FLOW-FIELD



LAGRANGIAN APPROACH



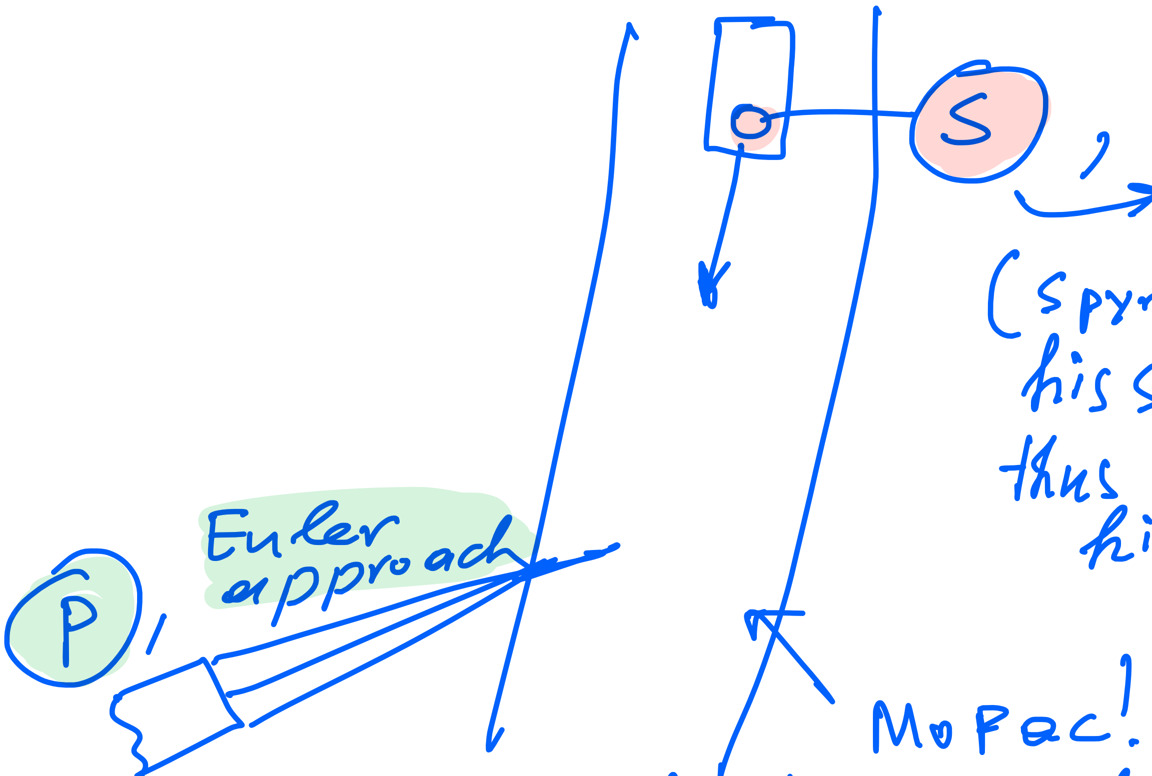
Follows the particles and records their velocities.

EULERIAN APPROACH

Does NOT follow the particles. It records velocities of particles passing through given locations

Most Common

EXAMPLE OF LAGRANGIAN VS. EULERIAN APPROACH

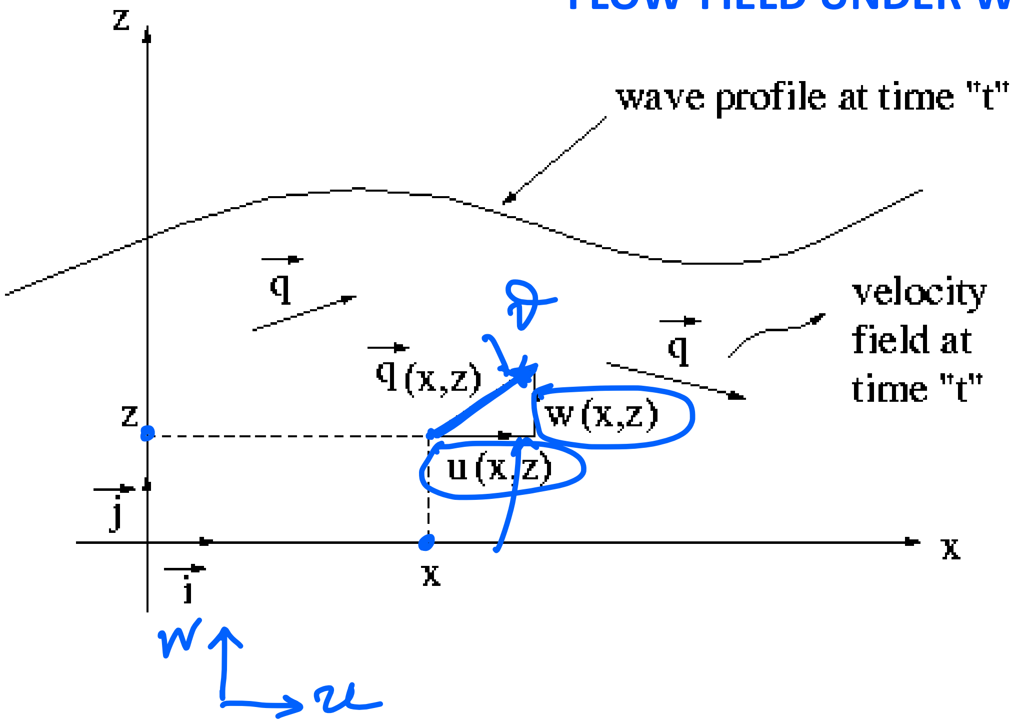


Lagrangian approach

(Spyros looks at his speedometer thus recording his speed)

[The Police record the velocities of vehicles ("particles") passing through a fixed location]

FLOW-FIELD UNDER WAVE SURFACE



Q: What laws should the fluid field follow?

$$|\vec{q}| = q = \sqrt{u^2 + w^2}$$

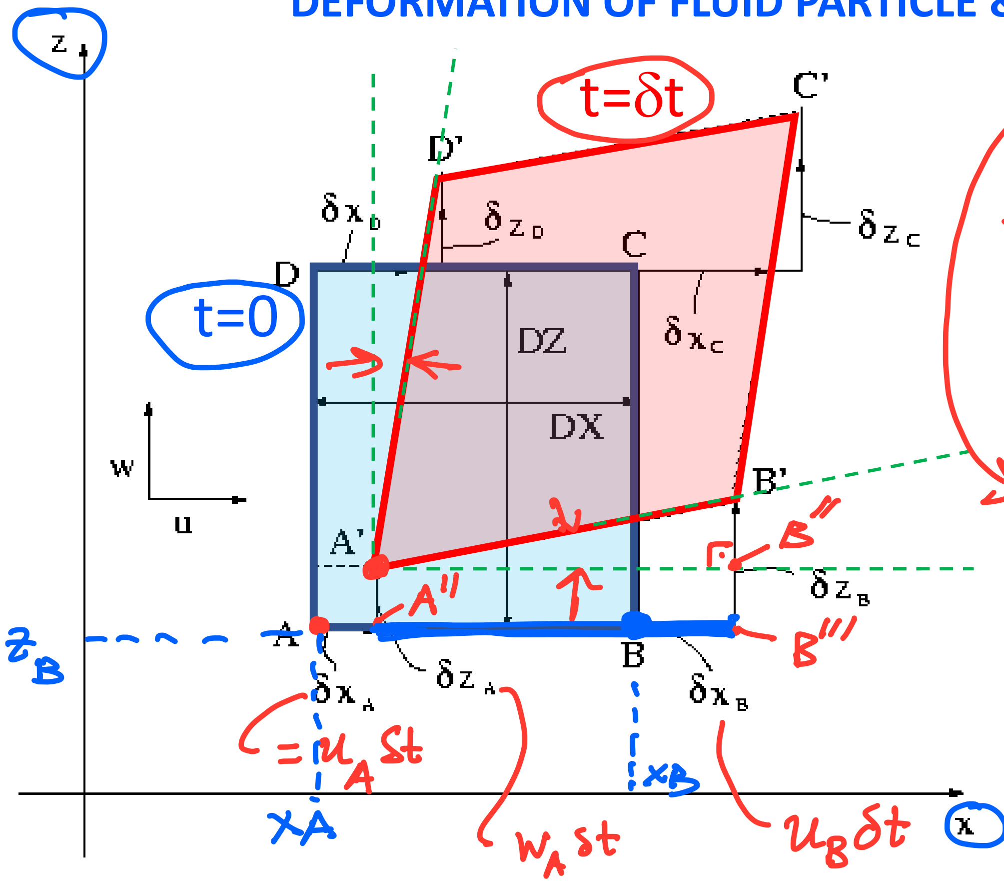
$$\tan\vartheta = \frac{w}{u}$$

A: The laws of physics!

1) conservation of mass!

2) Newton's Law

DEFORMATION OF FLUID PARTICLE & CONSERVATION OF MASS



$$m_{ABCD} = m_{A'B'C'D'}$$

Incompressible fluid

ρ (density) $= \dfrac{m}{\forall}$ is constant

$$\cancel{\rho}\, \forall_{ABCD} = \cancel{\rho}\, \forall_{A'B'C'D'}$$

$$DX \cdot DZ = (A'B')(A'D')$$

$$A'B' = A'B'' = A''B''' = \underbrace{AB'''}_{Dx + \underbrace{BB'''}_{u_B \delta t}} - \underbrace{AA''}_{u_A \delta t}$$

$$A'B' = Dx + u_B \delta t - u_A \delta t = Dx + (u_B - u_A)\delta t$$

$\forall$

$$u_B = u(x_B, z_B) = u(x_A + Dx, z_A) =$$

$$= \underbrace{u(x_A, z_A)} + \frac{\partial u}{\partial x} \underset{Dx}{\underline{\Delta x}} + \frac{\partial u}{\partial z} \underset{=0}{\cancel{\Delta z}} =$$

$$= u_A + \frac{\partial u}{\partial x} Dx \Rightarrow (u_B - u_A) = \frac{\partial u}{\partial x} Dx$$

$$A'B' = Dx + \frac{\partial u}{\partial x} Dx\, \delta t = Dx\left[1 + \frac{\partial u}{\partial x} \delta t\right]$$

$$A'D' = Dz\left[1 + \frac{\partial w}{\partial z} \delta t\right]$$

$$\cancel{Dx} \cdot \cancel{Dz} = \cancel{Dx}\left[1 + \frac{\partial u}{\partial x} \delta t\right] \cdot \cancel{Dz}\left[1 + \frac{\partial w}{\partial z} \delta t\right]$$

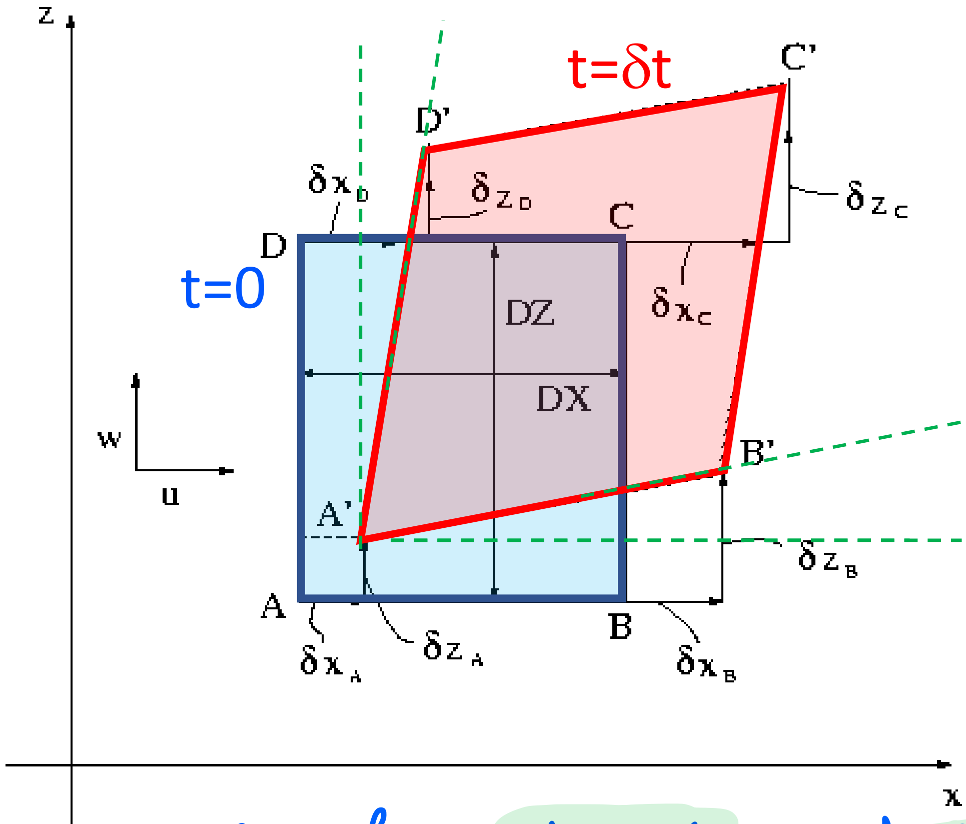
$$1 = \left[1 + \frac{\partial u}{\partial x} \delta t\right]\left[1 + \frac{\partial w}{\partial z} \delta t\right]$$

HOT

$$\cancel{1} = \cancel{1} + \frac{\partial u}{\partial x} \delta t + \frac{\partial w}{\partial z} \delta t + \cancel{\frac{\partial u}{\partial x} \frac{\partial w}{\partial z} (\delta t)^2}$$

$$0 = \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \delta t \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

DEFORMATION OF FLUID PARTICLE & CONSERVATION OF MASS



$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0}$$

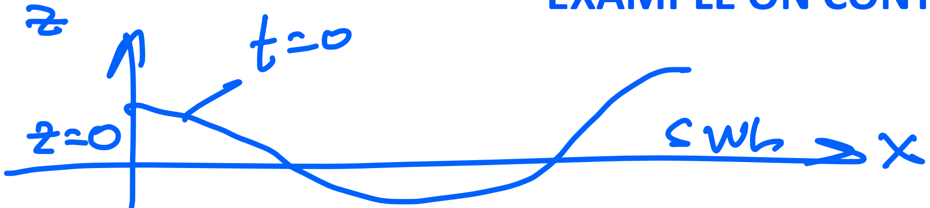
Continuity equation must be valid everywhere in the flowfield and at ALL times.

- valid for steady and unsteady flows
- valid for inviscid and viscous flows

In 3-D: $\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$

CONTINUITY EQUATION FOR INCOMPRESSIBLE FLUIDS

EXAMPLE ON CONTINUITY EQUATION



Flowfield under wave can be expressed as:

- $u = e^{\lambda z}\cos(kx)\,k$
- $w = \lambda e^{\lambda z}\sin(kx)$

(to be derived later)

Find λ so that this flowfield is physically meaningful

Continuity eqn. must be valid!

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = e^{\lambda z} k(-\sin(kx))k = -e^{\lambda z}k^2\sin(kx)$$

$$\frac{\partial w}{\partial z} = \lambda\sin(kx)\,e^{\lambda z}\lambda = \lambda^2 e^{\lambda z}\sin(kx)$$

$$-e^{\lambda z}k^2\sin(kx) + \lambda^2 e^{\lambda z}\sin(kx) = 0$$

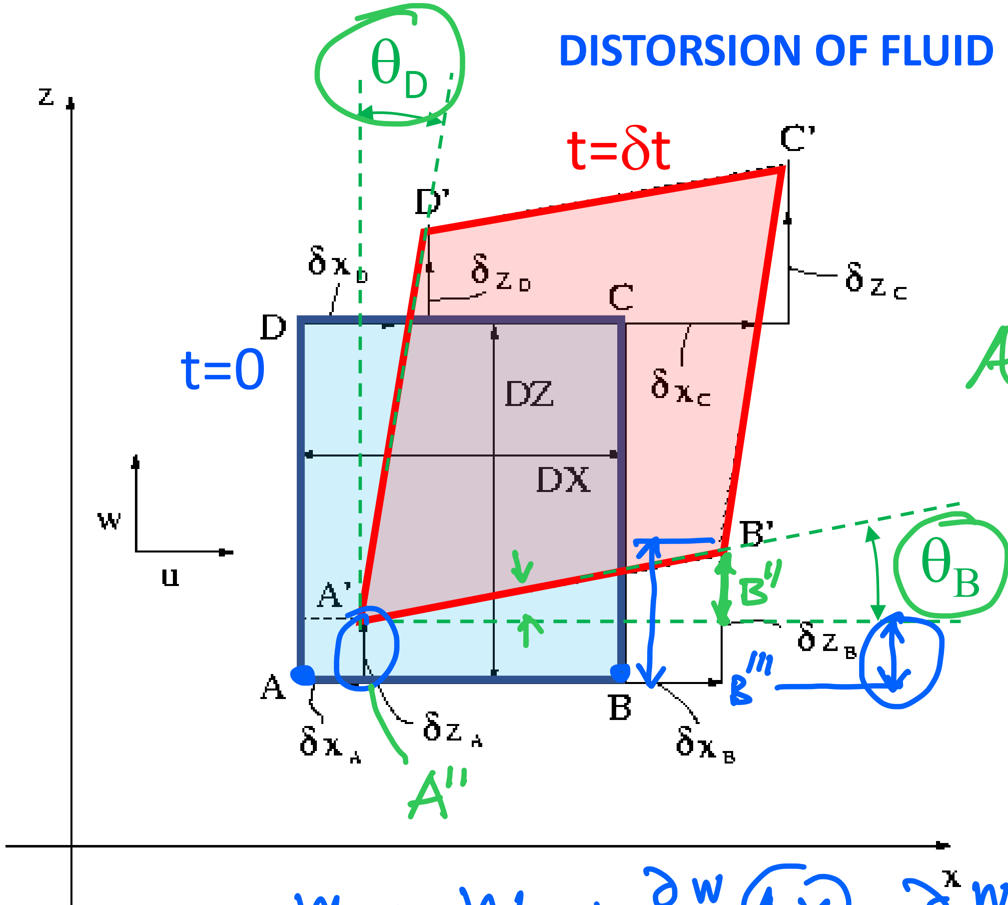
$$e^{\lambda z}\sin(kx)\left[-k^2+\lambda^2\right] = 0 \Rightarrow -k^2+\lambda^2 = 0 \quad \boxed{\lambda = k}$$

$$\Rightarrow \lambda^2 = k^2 \Rightarrow \lambda = +k, \ \cancel{-k}$$

Remember:

$$\frac{de^x}{dx} = e^x$$

DISTORSION OF FLUID PARTICLE



$$\vartheta_B = \tan\vartheta_B = \frac{B'B''}{A'B''}$$

$$A'B'' = A'B''' = Dx\left[1+\frac{\partial u}{\partial x}\delta t\right]$$

$$\boxed{B'B''} = \underbrace{B'B'''}_{w_B\,\delta t} - \underbrace{B''B'''}_{=\,w_A\,\delta t} = (w_B - w_A)\,\delta t = \boxed{\frac{\partial w}{\partial x}Dx\,\delta t}$$

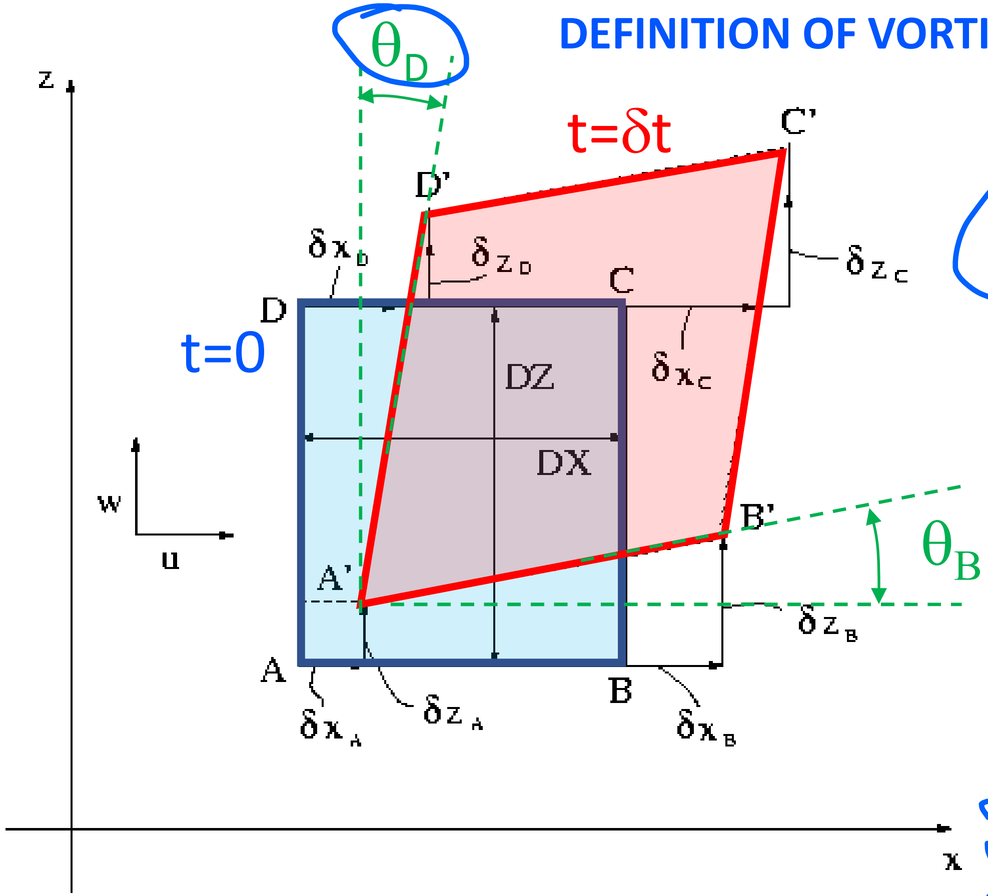
$$w_B = w_A + \frac{\partial w}{\partial x}\underbrace{\Delta x}_{=\,Dx} + \cancel{\frac{\partial w}{\partial z}\Delta z}_{=0}$$

Taylor in 2-D

$$\Rightarrow w_B - w_A = \frac{\partial w}{\partial x}Dx$$

$$\vartheta_B = \tan\vartheta_B = \frac{\frac{\partial w}{\partial x}\cancel{Dx}\,\delta t}{\cancel{Dx}\left[1+\frac{\partial u}{\partial x}\delta t\right]} = \frac{\frac{\partial w}{\partial x}\delta t}{1+\frac{\partial u}{\partial x}\delta t}$$

DEFINITION OF VORTICITY



remember $\frac{1}{1+a} \approx 1-a$ (a small)

$$\frac{\partial w}{\partial x}\delta t\left[1-\frac{\partial u}{\partial x}\delta t\right] = \frac{\partial w}{\partial x}\delta t - \cancel{\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}(\delta t)^2}$$

HOT

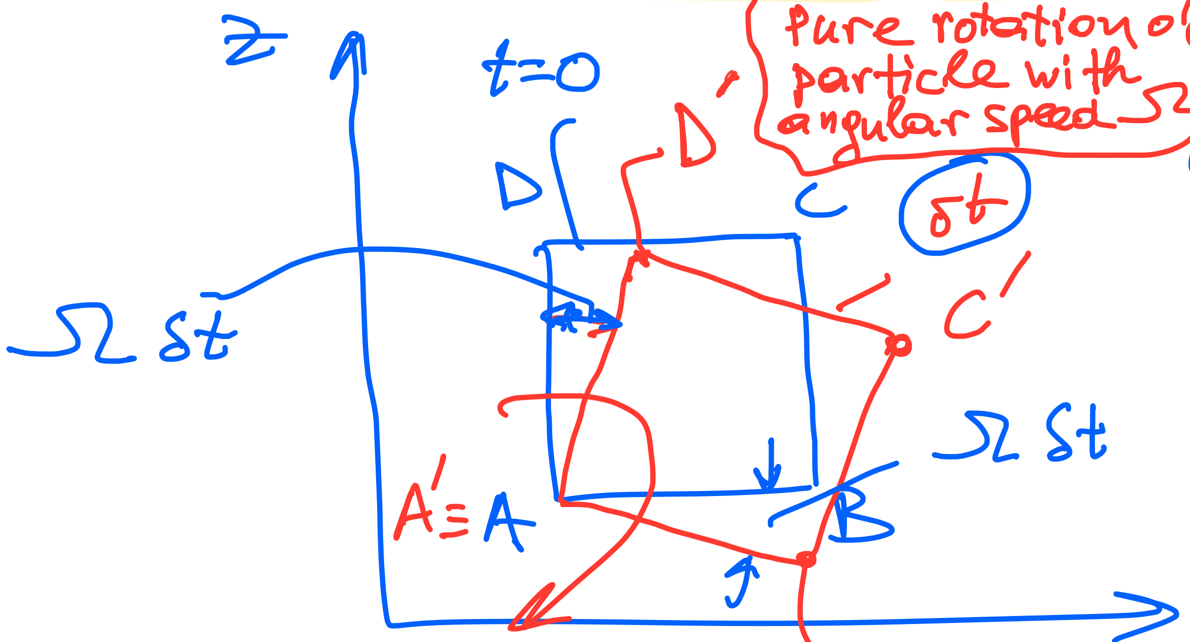
$$\Rightarrow \boxed{\vartheta_B = \frac{\partial w}{\partial x}\delta t}$$

Similarly $\boxed{\vartheta_D = \frac{\partial u}{\partial z}\delta t}$

Definition of vorticity:

$$\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\vartheta_D}{\delta t} - \frac{\vartheta_B}{\delta t} = \frac{\vartheta_D - \vartheta_B}{\delta t}$$

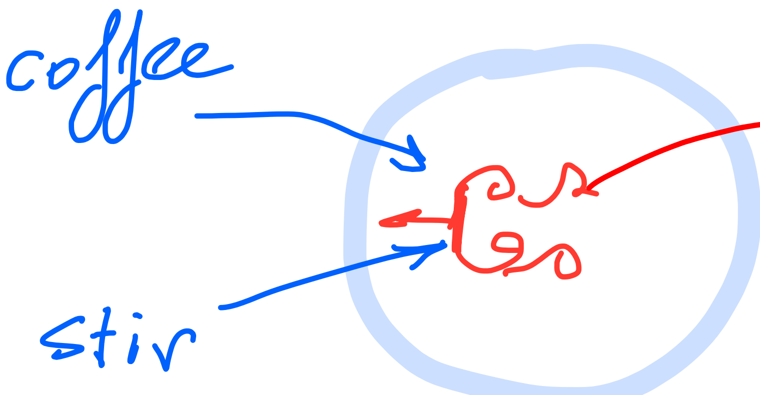
PHYSICAL MEANING OF VORTICITY



$$\omega = \frac{\vartheta_D - \vartheta_B}{\delta t} = \frac{\Omega\,\delta t - (-\Omega\,\delta t)}{\delta t} = 2\Omega$$

angular velocity or spin of particle Ω

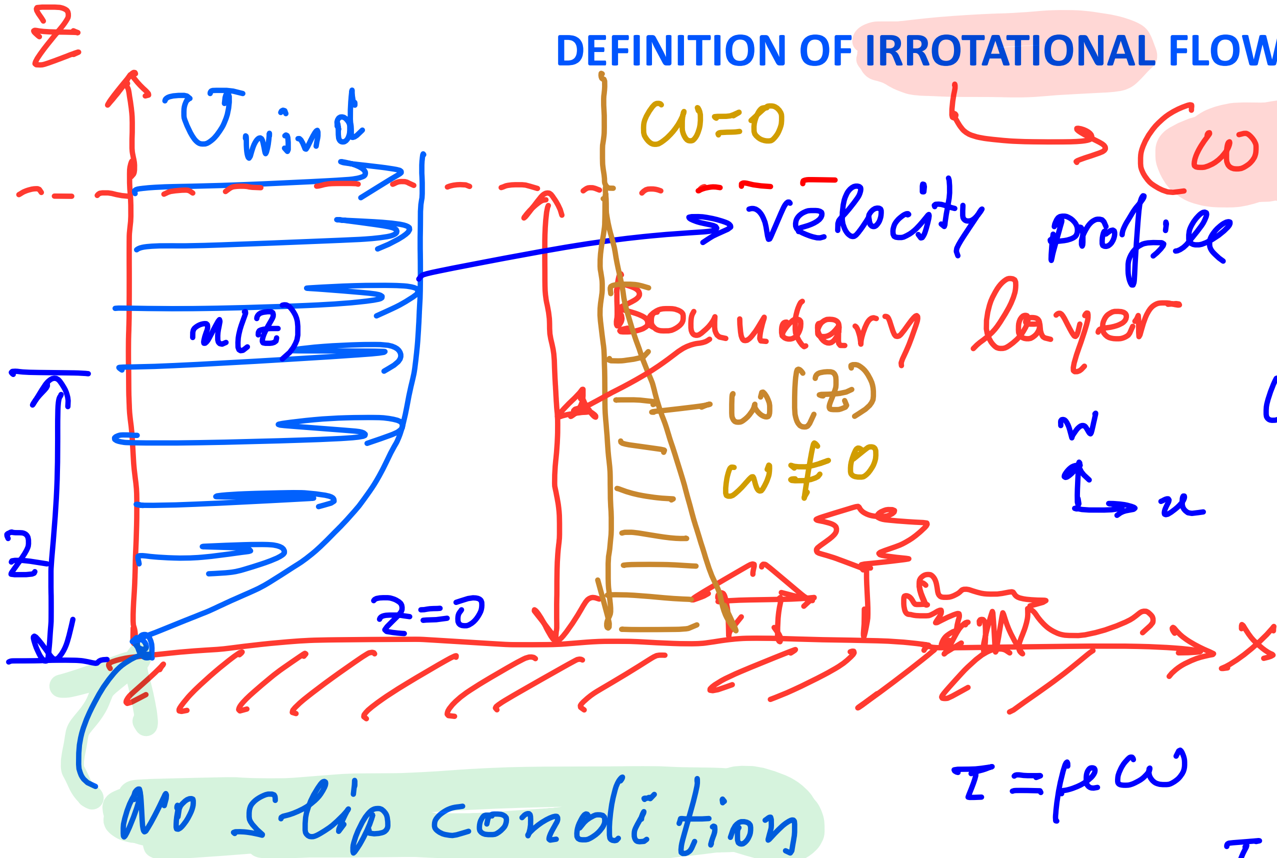
Vorticity ← $\boxed{\omega = 2\Omega}$ → spin of particle



eddies or swirl or vortices (other names for vorticity)

DEFINITION OF IRROTATIONAL FLOW

$(\omega = 0)$



w
u

$$\omega = \frac{\partial u}{\partial z} - \cancel{\frac{\partial w}{\partial x}}$$

≈ 0

$$\tau = \mu \frac{\partial u}{\partial z}$$

dynamic viscosity

$\tau = \mu \omega$

or $\omega = \dfrac{\tau}{\mu}$

τ and ω are most important (≠0) close to solid boundaries!