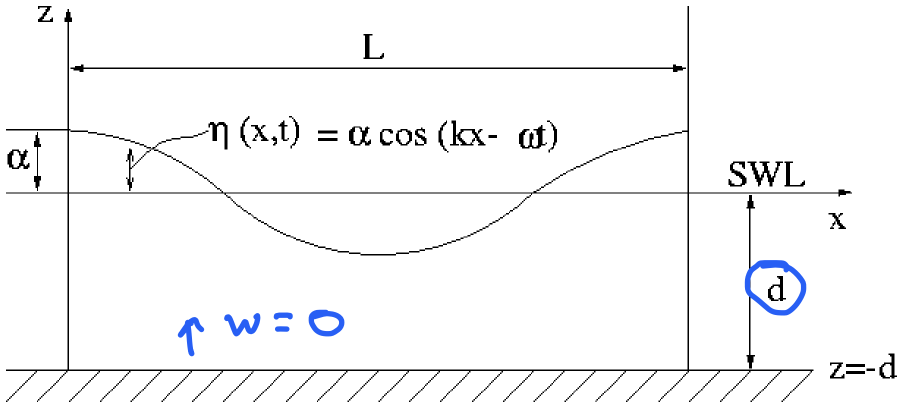
LINEAR WAVE THEORY – FINITE DEPTH WATER



$$k = \frac{2\pi}{L}$$

$$\omega = \frac{2\pi}{T}$$

$$\nabla^2 \phi = 0$$

Remember in deep H_2O

$$\phi = A e^{kz} \sin(kx - \omega t)$$

$$\nabla^2 \phi = 0 \quad ; \quad A = \frac{a\omega}{k}$$

$$\varphi(x,z,t) = \left(Ae^{kz} + Be^{-kz}\right)\sin(kx - \omega t) = \qquad (62)$$

$$= A e^{kz} \sin(kx - \omega t) + B e^{-kz} \sin(kx - \omega t)$$

$$\varphi(x,z,t) = \frac{ga}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \sin(kx - \omega t) \qquad \text{Finite Depth} \qquad (63)$$

A, B can be determined from applying kbc at the sea-floor and the free-surface

LINEAR WAVE THEORY – FINITE DEPTH WATER – DISPERSION RELATIONSHIP

From the dynamic bc (dbc) at the free-surface ($p = 0$)

DISPERSION RELATIONSHIP $$\boxed{\frac{\omega^2}{k\tanh(kd)} = g}$$

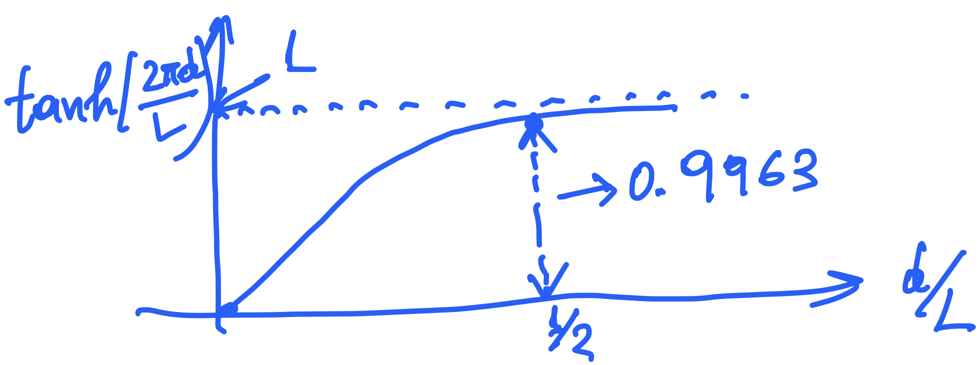
Remember for deep H_2O $$\boxed{\frac{\omega^2}{k} = g}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

as $x \to \infty$ $\boxed{\tanh x \to 1}$

as $x \to 0$ $e^x \approx 1 + x$ as $x \to 0$; $\tanh x \sim \frac{(1+x) - (1-x)}{(1+x) + (1-x)} = \frac{2x}{2}$

$\boxed{\tanh x \sim x}$ as $x \to 0$



LINEAR WAVE THEORY – DISPERSION RELATIONSHIP - EXAMPLE

$$\frac{\omega^2}{k\tanh(kd)} = g \sim \frac{\left(\frac{2\pi}{T}\right)^2}{\frac{2\pi}{L}\tanh\left(\frac{2\pi d}{L}\right)} = g \Rightarrow$$

$$\Rightarrow \boxed{L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

If d and L are given then we can solve for T!

$\hookrightarrow L_0$ deep H_2O wave length

$$L = L_0 \tanh\left(\frac{2\pi d}{L}\right) \sim$$

$$\sim \boxed{\frac{d}{L_0} = \frac{d}{L}\tanh\left(\frac{2\pi d}{L}\right)}$$

Example: Find L for T= 7sec at d= 8m

$$L_0 = \frac{gT^2}{2\pi} = \frac{9.81\times 7^2}{2\pi} = 76.5m \Rightarrow \frac{d}{L_0} = \frac{8}{76.5} = 0.1046 \xrightarrow[C-1]{\text{Table}} \frac{d}{L} = 0.1449$$

$$\Rightarrow L = d/0.1449 = \boxed{55.2m}$$

LINEAR WAVE THEORY – FINITE DEPTH WATER

CE358 (Kinnas) – Fall 1998 – UT Austin

34b

Related to Example Problem under section: "How to find L for given T and d?"

$$L_o = \frac{gT^2}{2\pi}$$

Table C-1. Continued.

d/L_o	d/L	$2\pi d/L$	TANH $2\pi d/L$	SINH $2\pi d/L$	COSH $2\pi d/L$	H/H_o'	K	$4\pi d/L$	SINH $4\pi d/L$	COSH $4\pi d/L$	n	C_G/C_o	%
.09000	.1322	.8306	.6808	.9295	1.3653	.9422	.7324	1.661	2.538	2.728	.8273	.5632	10.65
.09100	.1331	.8363	.6838	.9372	1.3706	.9411	.7296	1.672	2.568	2.756	.8255	.5645	10.55
.09200	.1340	.8420	.6868	.9450	1.3759	.9401	.7268	1.684	2.599	2.785	.8238	.5658	10.46
.09300	.1349	.8474	.6897	.9525	1.3810	.9391	.7241	1.695	2.630	2.814	.8221	.5670	10.37
.09400	.1357	.8528	.6925	.9600	1.3862	.9381	.7214	1.706	2.662	2.843	.8204	.5682	10.29
.09500	.1366	.8583	.6953	.9677	1.3917	.9371	.7186	1.717	2.693	2.873	.8187	.5693	10.21
.09600	.1375	.8639	.6982	.9755	1.3970	.9362	.7158	1.728	2.726	2.903	.8170	.5704	10.12
.09700	.1384	.8694	.7011	.9832	1.4023	.9353	.7131	1.739	2.757	2.933	.8153	.5716	10.04
.09800	.1392	.8749	.7039	.9908	1.4077	.9344	.7104	1.750	2.790	2.963	.8136	.5727	9.962
.09900	.1401	.8803	.7066	.9985	1.4131	.9335	.7076	1.761	2.822	2.994	.8120	.5737	9.884
.1000	.1410	.8858	.7093	1.006	1.4187	.9327	.7049	1.772	2.855	3.025	.8103	.5747	9.808
.1010	.1419	.8913	.7120	1.014	1.4242	.9319	.7022	1.783	2.888	3.057	.8086	.5757	9.734
.1020	.1427	.8967	.7147	1.022	1.4297	.9311	.6994	1.793	2.922	3.088	.8069	.5766	9.661
.1030	.1436	.9023	.7173	1.030	1.4354	.9304	.6967	1.805	2.956	3.121	.8052	.5776	9.590
.1040	.1445	.9076	.7200	1.037	1.4410	.9297	.6940	1.815	2.990	3.153	.8036	.5785	9.519
.1050	.1453	.9130	.7226	1.045	1.4465	.9290	.6913	1.826	3.024	3.185	.8019	.5794	9.451
.1060	.1462	.9184	.7252	1.053	1.4523	.9282	.6886	1.837	3.059	3.218	.8003	.5803	9.384
.1070	.1470	.9239	.7277	1.061	1.4580	.9276	.6859	1.848	3.094	3.251	.7986	.5812	9.318
.1080	.1479	.9293	.7303	1.069	1.4638	.9269	.6833	1.858	3.128	3.284	.7970	.5820	9.254
.1090	.1488	.9343	.7327	1.076	1.4692	.9263	.6806	1.869	3.164	3.319	.7954	.5828	9.191
.1100	.1496	.9400	.7352	1.085	1.4752	.9257	.6779	1.880	3.201	3.353	.7937	.5836	9.129
.1110	.1505	.9456	.7377	1.093	1.4814	.9251	.6752	1.891	3.237	3.388	.7920	.5843	9.068
.1120	.1513	.9508	.7402	1.101	1.4871	.9245	.6725	1.902	3.274	3.423	.7904	.5850	9.009
.1130	.1522	.9563	.7426	1.109	1.4932	.9239	.6697	1.913	3.312	3.459	.7888	.5857	8.950
.1140	.1530	.9616	.7450	1.117	1.4990	.9234	.6671	1.923	3.348	3.494	.7872	.5864	8.891

$d/L_o = 0.104$

LINEAR WAVE THEORY – PARTICLE VELOCITIES

$$\varphi(x,z,t) = \frac{ga}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \sin(kx - \omega t) \qquad \text{Finite Depth} \tag{63}$$

$$u = \frac{\partial \phi}{\partial x} = \frac{ga}{\omega} \frac{\cosh[k(z+d)]}{\cosh[kd]} \cos(kx - \omega t) \cdot k$$

$$\left(\frac{d \cosh x}{dx} = \sinh x \right)$$

$$w = \frac{\partial \phi}{\partial z} = \ldots$$

$$\frac{g \frac{H}{2} \cancel{2\pi}}{L \frac{\cancel{2\pi}}{T}} = \frac{gHT}{2L} \rightarrow \text{same as in Fig. 2-6}$$

$$u(x,z,t) = \frac{\partial \varphi}{\partial x} = \frac{gak}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \cdot \cos(kx - \omega t) \tag{79}$$

$$w(x,z,t) = \frac{\partial \varphi}{\partial z} = \frac{gak}{\omega} \cdot \frac{\sinh[k(z+d)]}{\cosh(kd)} \cdot \sin(kx - \omega t) \tag{80}$$

LINEAR WAVE THEORY – ACCELERATIONS

$$u(x,z,t) = \frac{\partial \varphi}{\partial x} = \frac{gak}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \cdot \cos(kx - \omega t) \qquad (79)$$

$$w(x,z,t) = \frac{\partial \varphi}{\partial z} = \frac{gak}{\omega} \cdot \frac{\sinh[k(z+d)]}{\cosh(kd)} \cdot \sin(kx - \omega t) \qquad (80)$$

$$a_x = \underbrace{\frac{\partial u}{\partial t}}_{\text{unsteady terms}} + \underbrace{\cancel{u \frac{\partial u}{\partial x}} + \cancel{w \frac{\partial u}{\partial z}}}_{\text{convective terms}}$$

$\sim a$ $\quad \sim a \quad \sim a \quad \sim a$

$\sim a^2 \qquad \sim a^2$

H.O.T.

(negligible)

$$a_x = \frac{\partial u}{\partial t} = gak \frac{\cosh[k(z+d)]}{\cosh(kd)} \cdot \sin(kx - \omega t) \qquad (81)$$

$$a_z = \frac{\partial w}{\partial t} = -gak \frac{\sinh[k(z+d)]}{\cosh(kd)} \cdot \cos(kx - \omega t) \qquad (82)$$

$\theta = kx - \omega t$

RELATIVE DEPTH	SHALLOW WATER $\frac{d}{L} < \frac{1}{25} = 0.04$	TRANSITIONAL WATER $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	DEEP WATER $\frac{d}{L} > \frac{1}{2}$
1. Wave profile	Same As →	$\eta = \frac{H}{2} \cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	← Same As
2. Wave celerity ✓	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_o = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength ✓	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_o = \frac{gT^2}{2\pi} = C_o T$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right] \cdot C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water Particle Velocity (a) Horizontal u	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$u = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \cos\theta$
(b) Vertical w	$w = \frac{H\pi}{T}\left(1 + \frac{z}{d}\right)\sin\theta$	$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$w = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \sin\theta$
6. Water Particle Accelerations (a) Horizontal a_x	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}} \sin\theta$
(b) Vertical a_z	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1 + \frac{z}{d}\right)\cos\theta$	$a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}} \cos\theta$
7. Water Particle Displacements (a) Horizontal ξ	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$	$\xi = -\underbrace{\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}}_{=A}\sin\theta$	$\xi = -\frac{H}{2} e^{\frac{2\pi z}{L}} \sin\theta$
(b) Vertical ζ	$\zeta = \frac{H}{2}\left(1 + \frac{z}{d}\right)\cos\theta$	$\zeta = \underbrace{\frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}}_{=B}\cos\theta$	$\zeta = \frac{H}{2} e^{\frac{2\pi z}{L}} \cos\theta$
8. Subsurface Pressure	$p = \rho g(\eta - z)$	$p = \rho g \eta \underbrace{\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}}_{K_z} - \rho g z$	$p = \rho g \eta \underbrace{e^{\frac{2\pi z}{L}}}_{K_z} - \rho g z$

Figure 2-6. Summary of linear (Airy) wave theory--wave characteristics.

LINEAR WAVE THEORY – PARTICLE TRAJECTORIES

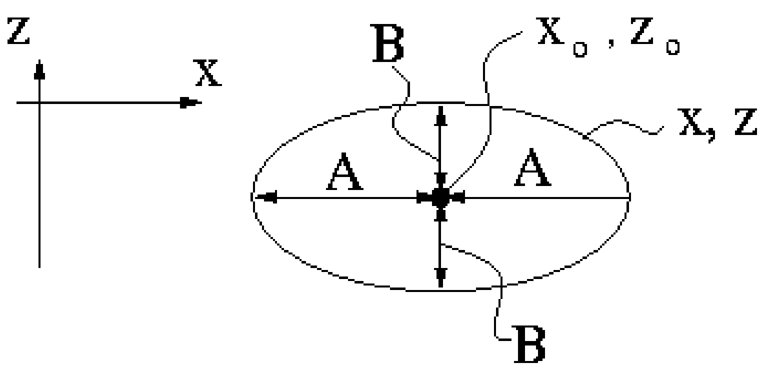
$$u(x,z,t) = \frac{\partial \varphi}{\partial x} = \frac{gak}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \cdot \cos(kx - \omega t) \tag{79}$$

$$w(x,z,t) = \frac{\partial \varphi}{\partial z} = \frac{gak}{\omega} \cdot \frac{\sinh[k(z+d)]}{\cosh(kd)} \cdot \sin(kx - \omega t) \tag{80}$$

$$u(x,z,t) \approx u(x_0, z_0, t) = \frac{dx}{dt} \tag{83}$$

$$w(x,z,t) \approx w(x_0, z_0, t) = \frac{dz}{dt} \tag{84}$$

by integrating these in time we will get elliptical trajectories



→ $$A = a \cdot \frac{\cosh[k(z_0+d)]}{\sinh(kd)}$$

→ $$B = a \cdot \frac{\sinh[k(z_0+d)]}{\sinh(kd)}$$

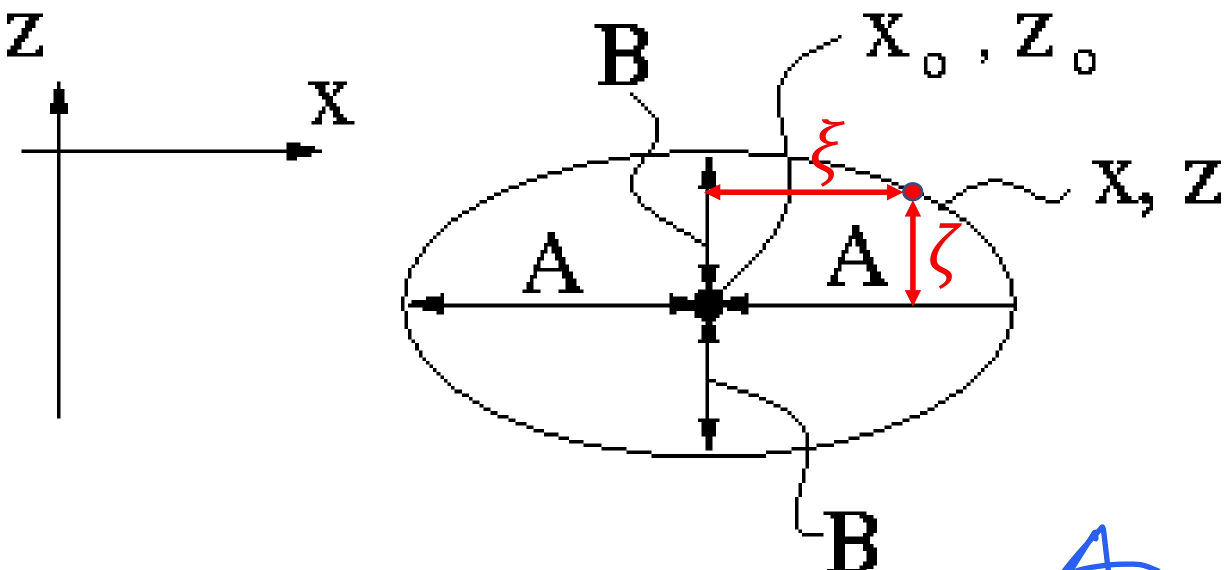
Note at the sea-floor at $z=-d \Rightarrow B=0$ ✓ $(\sinh 0 = 0)$

at the free-surface (at $z \simeq 0$) $\rightsquigarrow B = a$ ✓

LINEAR WAVE THEORY – PARTICLE TRAJECTORIES

$$A = a \cdot \frac{\cosh[k(z_0 + d)]}{\sinh(kd)} \tag{85}$$

$$B = a \cdot \frac{\sinh[k(z_0 + d)]}{\sinh(kd)} \tag{86}$$



$$\xi = -\underbrace{\frac{H}{2}\,\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}}_{A}\sin\theta$$

$$\zeta = \underbrace{\frac{H}{2}\,\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}}_{B}\cos\theta$$

$$\frac{\xi^2}{A^2} + \frac{\zeta^2}{B^2} = 1$$

LINEAR WAVE THEORY – PRESSURES

$$\varphi(x,z,t) = \frac{ga}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \sin(kx - \omega t) \qquad \text{Finite Depth} \qquad (63)$$

Bernoulli equ:

$$\rho \frac{\partial \phi}{\partial t} + p + \frac{\rho q^2}{2} + \rho g z = 0$$

$$\Rightarrow p = -\rho \frac{\partial \phi}{\partial t} - \cancel{\frac{\rho q^2}{2}} - \rho g z$$

H.O.T.

$$\frac{\partial \phi}{\partial t} = \frac{ga}{\cancel{\omega}} \frac{\cosh[k(z+d)]}{\cosh[kd]} \cos(kx - \omega t)(-\cancel{\omega})$$

η

$$p = \rho g \eta \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$$

gage pressure

wave pressure

hydrostatic term in the absence of waves

LINEAR WAVE THEORY – PRESSURES

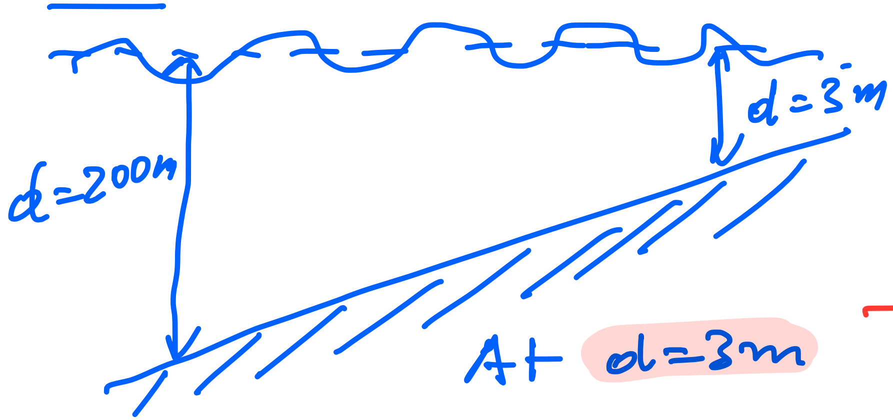
$$p = \rho g \eta \underbrace{\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}}_{= K_z} - \rho g z$$

$= K_z$ pressure response factor

Example problem 1 on p. 2-11 of SPM

A 10 sec wave propagates from a depth of 200m to a depth of 3m. Find the length and the speed of the wave at d=200 m and d=3m

T=10sec



At d=200m

$\boxed{L_0} = \frac{gT^2}{2\pi} = \boxed{156\,m}$ deep H_2O! $\left(d > \frac{L}{2}\right)$

$\boxed{C} = C_0 = \frac{L_0}{T} = \boxed{15.6\,\frac{m}{s}}$

At d=3m

T has to stay the same!

$L_0 = 156\,m$ $\quad \frac{d}{L_0} = \frac{3}{156} = 0.0192 \xrightarrow[C-1]{} \frac{d}{L} = 0.05641 \rightsquigarrow \boxed{L = 53.2\,m}$

$C = L/T = 5.32\,m/s$

LINEAR WAVE THEORY – PRESSURES - EXAMPLE

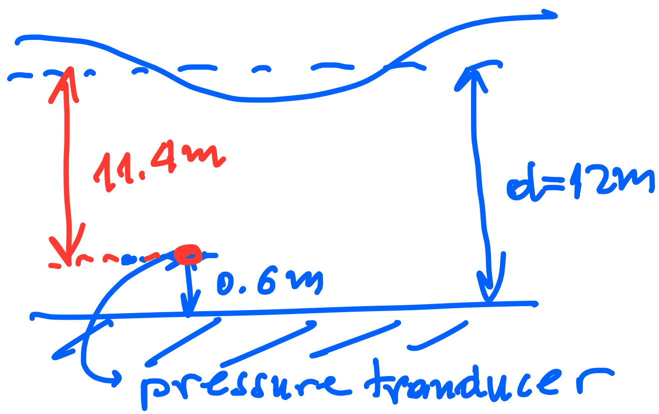
$$p = \rho g \eta \underbrace{\frac{\cosh[2\pi(\overset{-11.6m}{z} + \overset{12m}{d})/L]}{\cosh(2\pi \underset{12m}{d}/L)}}_{=K_z} - \rho g z$$

Example Problem 5 on p. 2-22 of SPM

pressure transducer measures a maximum pressure (gage) of $124 \frac{kN}{m^2}$

Frequency of the waves is $f = 0.0666$ cycles/sec (or Hz)

Find H



$$P_{max} = \rho g a K_z - \rho g z \rightarrow z = -11.4m$$

$$a = \frac{H}{2}$$

$$\rightarrow P_{max} + \rho g z = \rho g \frac{H}{2} K_z \rightarrow$$

$$\rightarrow H = \left(\frac{P_{max} + \rho g z}{\rho g K_z}\right) \times 2$$

$$T = \frac{1}{f} = \frac{1}{0.0666} = 15 \text{ sec}$$

$$L_0 = \frac{gT^2}{2\pi} = 351m \rightarrow \frac{d}{L_0} = \frac{12}{351} = 0.0342 \xRightarrow[C-1]{Table} \frac{d}{L} = 0.07651$$

$$\Rightarrow L = 156.8m$$

LINEAR WAVE THEORY – PRESSURES - EXAMPLE

$$p = \rho g \eta \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$$

$$\underline{K_z} = \frac{\cosh[2\pi(-11.4+12)/156.8]}{\cosh[2\pi \times 12/156.8]} = \underline{0.8949}$$

$$\Rightarrow \boxed{H = 1.04\ m}$$

LINEAR WAVE THEORY – SHALLOW WATER - APPROXIMATIONS as d/L -->0/PRACTICAL LIMIT

$$e^{x} \approx 1+x \quad \text{for small } x \quad (x>0)$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2} \longrightarrow \frac{(1+x)-(1-x)}{2} = x$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2} \longrightarrow \frac{(1+x)+(1-x)}{2} = 1$$

$$\tanh x = \frac{\sinh x}{\cosh x} \longrightarrow x$$

In the case of wave speed:

$$C = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right) \Longrightarrow C = \sqrt{gd}$$

$$\tanh(x) \to x$$

$$\tanh\left(\frac{2\pi d}{L}\right) \to \frac{2\pi d}{L}$$

$$c = \frac{gT}{\cancel{2\pi}} \cdot \frac{\cancel{2\pi} d}{L} \Rightarrow \frac{g d T}{L} = \frac{g d}{c} = c$$

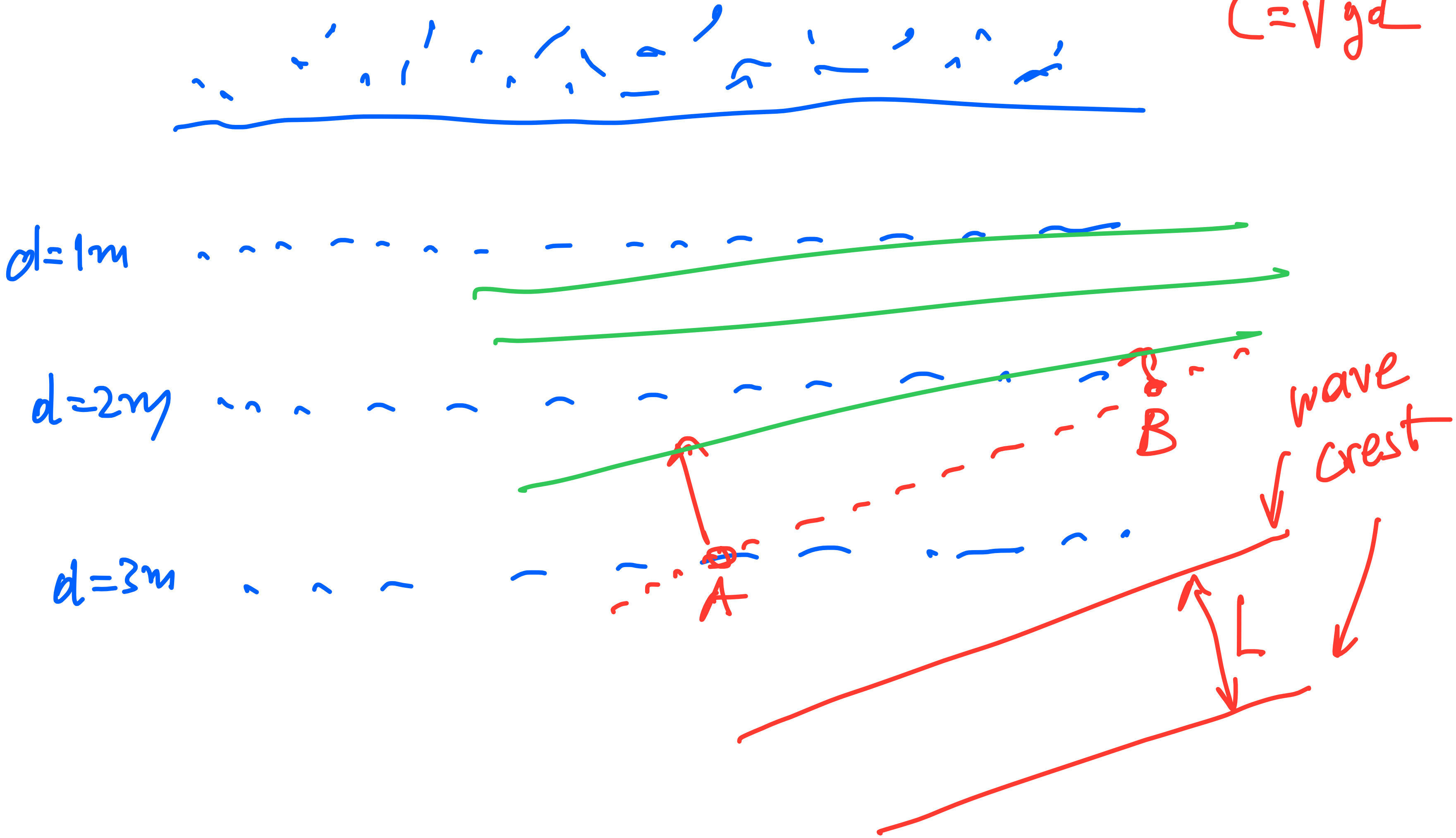
$$c^2 = g d \Rightarrow \boxed{\sqrt{g d} = c}$$

LINEAR WAVE THEORY – SHALLOW WATER - WAVE SPEED

$$C = \frac{gT}{2\pi} \tanh(\frac{2\pi d}{L}) \qquad \text{(wave speed for finite depth water)} \qquad (90)$$

LINEAR WAVE THEORY – SHALLOW WATER - WAVE SPEED – APPLICATION TO REFRACTION

$$C=\sqrt{gd}$$



ALWAYS VALID

RELATIVE DEPTH	SHALLOW WATER $\frac{d}{L} < \frac{1}{25}$	TRANSITIONAL WATER $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	DEEP WATER $\frac{d}{L} > \frac{1}{2}$
1. Wave profile	Same As →	$\eta = \frac{H}{2}\cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	← Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_o = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_o = \frac{gT^2}{2\pi} = C_oT$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right] \cdot C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water Particle Velocity (a) Horizontal	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$ u_{max}	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$u = \frac{\pi H}{T}e^{\frac{2\pi z}{L}}\cos\theta$
(b) Vertical	$w = \frac{H\pi}{T}\left(1 + \frac{z}{d}\right)\sin\theta$	$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$w = \frac{\pi H}{T}e^{\frac{2\pi z}{L}}\sin\theta$
6. Water Particle Accelerations (a) Horizontal	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}}\sin\theta$
(b) Vertical	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1 + \frac{z}{d}\right)\cos\theta$	$a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}}\cos\theta$
7. Water Particle Displacements (a) Horizontal	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$ A	$\xi = -\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\sin\theta$	$\xi = -\frac{H}{2}e^{\frac{2\pi z}{L}}\sin\theta$
(b) Vertical	$\zeta = \frac{H}{2}\left(1 + \frac{z}{d}\right)\cos\theta$ B	$\zeta = \frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\cos\theta$	$\zeta = \frac{H}{2}e^{\frac{2\pi z}{L}}\cos\theta$
8. Subsurface Pressure	$p = \rho g(\eta - z)$	$p = \rho g\eta\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho gz$	$p = \rho g\eta e^{\frac{2\pi z}{L}} - \rho gz$

Figure 2-6. Summary of linear (Airy) wave theory--wave characteristics.

gage press.

wave press.

hydrostatic when no waves

LINEAR WAVE THEORY – SHALLOW WATER - VELOCITIES

$$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$$

≈ 1

≈ 1

$$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$$

$$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$$

independent of z!

$$w = \frac{H\pi}{T}\left(1+\frac{z}{d}\right)\sin\theta$$

$$u = \frac{H}{2}\frac{gT}{L} = \frac{H}{2}\frac{g}{C} = \frac{H}{2}\frac{g}{\sqrt{gd}} =$$

LINEAR WAVE THEORY – SHALLOW WATER - ACCELERATIONS

$$a_x = \frac{g\pi H}{L}\,\frac{\cosh\left[2\pi(z+d)/L\right]}{\cosh(2\pi d/L)}\,\sin\theta$$

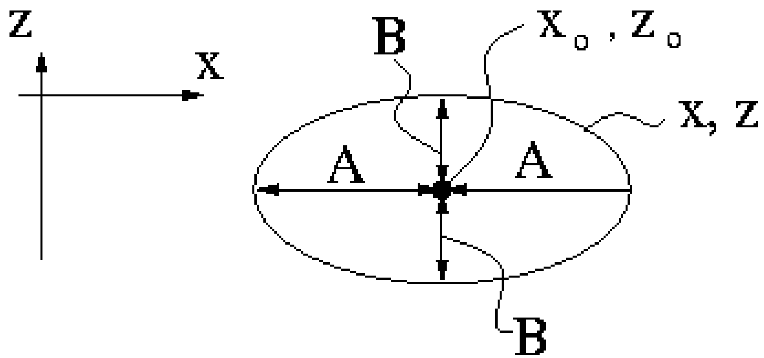
$$a_z = -\frac{g\pi H}{L}\,\frac{\sinh\left[2\pi(z+d)/L\right]}{\cosh(2\pi d/L)}\,\cos\theta$$

$$\Rightarrow$$

$$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\,\sin\theta$$

$$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1+\frac{z}{d}\right)\cos\theta$$

LINEAR WAVE THEORY – SHALLOW WATER – PARTICLE TRAJECTORIES



$$A = a \cdot \frac{\cosh[k(z_0 + d)]}{\sinh(kd)}$$

$$B = a \cdot \frac{\sinh[k(z_0 + d)]}{\sinh(kd)}$$

for shallow water: $A = a \dfrac{1}{kd} =$

$(\sinh x \to x)$

$$= \frac{a}{kd} = \frac{(H/2)}{\left(\frac{2\pi}{L}\right)d} = \frac{HT}{4\pi}\sqrt{\frac{g}{d}}$$

$$L = \underbrace{\sqrt{gd}}_{c} \cdot T$$

LINEAR WAVE THEORY – SHALLOW WATER – PRESSURES

$$p = \rho g \eta \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$$

$\cosh[2\pi(z+d)/L] \rightarrow 1$

$\cosh(2\pi d/L) \rightarrow 1$

For shallow :

$$\boxed{p = \rho g \eta - \rho g z} = \rho g(\eta - z)$$

RELATIVE DEPTH	SHALLOW WATER $\frac{d}{L} < \frac{1}{25}$	TRANSITIONAL WATER $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	DEEP WATER $\frac{d}{L} > \frac{1}{2}$
1. Wave profile	Same As →	$\eta = \frac{H}{2} \cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	← Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_0 = \frac{gT^2}{2\pi} = C_0 T$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right]\cdot C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water Particle Velocity (a) Horizontal	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$u = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \cos\theta$
(b) Vertical	$w = \frac{H\pi}{T}\left(1 + \frac{z}{d}\right)\sin\theta$	$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$w = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \sin\theta$
6. Water Particle Accelerations (a) Horizontal	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}} \sin\theta$
(b) Vertical	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1 + \frac{z}{d}\right)\cos\theta$	$a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}} \cos\theta$
7. Water Particle Displacements (a) Horizontal	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$ A	$\xi = -\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\sin\theta$	$\xi = -\frac{H}{2} e^{\frac{2\pi z}{L}} \sin\theta$
(b) Vertical	$\zeta = \frac{H}{2}\left(1 + \frac{z}{d}\right)\cos\theta$ B	$\zeta = \frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\cos\theta$	$\zeta = \frac{H}{2} e^{\frac{2\pi z}{L}} \cos\theta$
8. Subsurface Pressure	$p = \rho g(\eta - z)$	$p = \rho g\eta\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$	$p = \rho g \eta e^{\frac{2\pi z}{L}} - \rho g z$

Figure 2-6. Summary of linear (Airy) wave theory--wave characteristics.

LINEAR WAVE THEORY – FINITE DEPTH WATER FORMULAS AS d/L →∞ (DEEP WATER)

$$C = \frac{L}{T} = \frac{gT}{2\pi} \tanh\left(\frac{2\pi d}{L}\right) \rightarrow 1$$

$$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$$

$$k$$

$$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$$

$$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$$

$$u = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \cos\theta$$

$$w = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \sin\theta$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \longrightarrow \frac{e^{x}}{2} \text{ as } x \to \infty$$

$$\frac{\frac{e^{k(z+d)}}{\cancel{2}}}{\frac{e^{kd}}{\cancel{2}}} = \frac{e^{k(z+d)}}{e^{kd}} = e^{kz}$$

LINEAR WAVE THEORY – FINITE DEPTH WATER FORMULAS AS d/L →∞ (DEEP WATER)

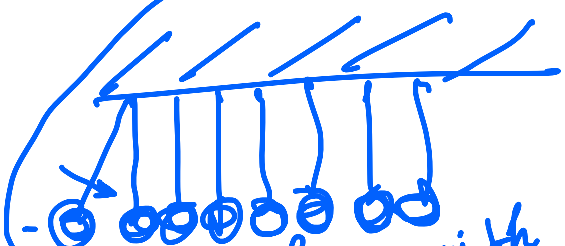
$$p = \rho g \eta \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z \qquad \Big| \qquad p = \rho g \eta e^{\frac{2\pi z}{L}} - \rho g z$$

$$\approx e^{kz}$$

For example for deep H_2O:

$$u = \frac{\pi H}{\cancel{T}} e^{kz} \cos\theta$$

$$\frac{u}{C} = \frac{(\quad)}{L/\cancel{T}} = \frac{\pi H}{L} e^{kz} \cos\theta$$



Analogy with problem where the motion of 1st sphere is limited but the disturbance moves far