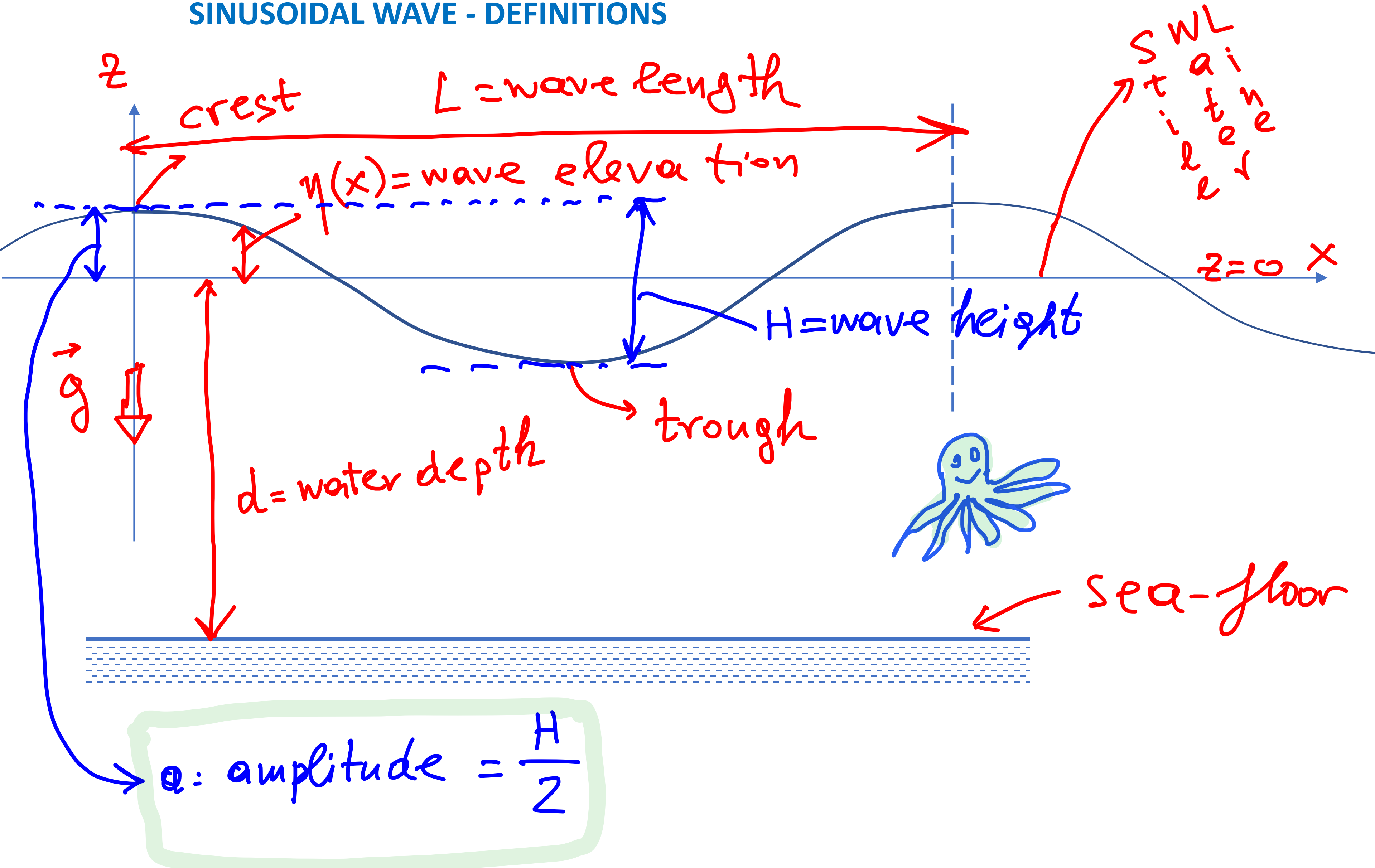
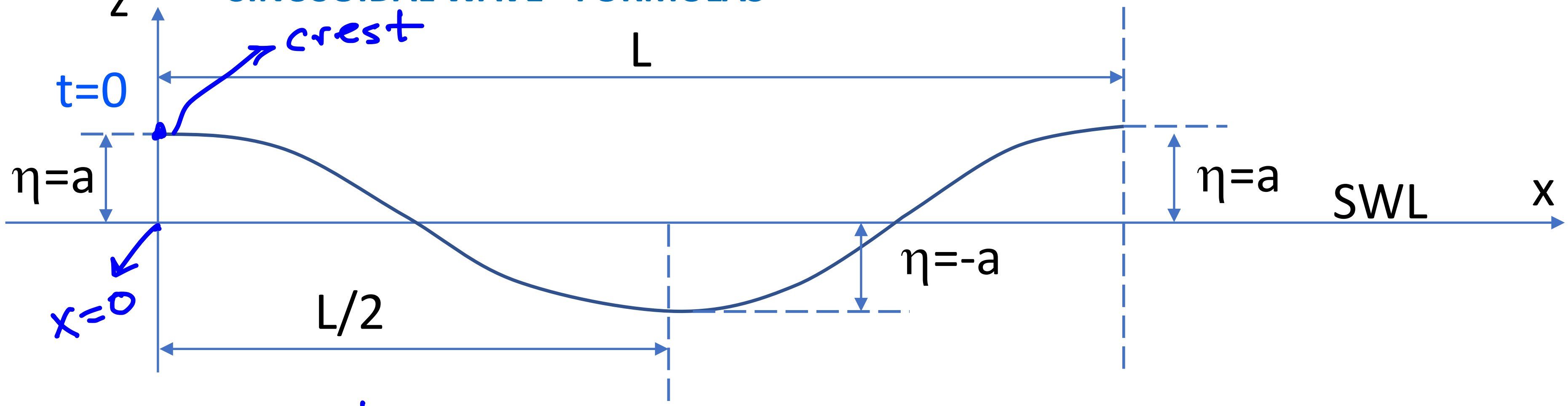
SINUSOIDAL WAVE - DEFINITIONS



SINUSOIDAL WAVE - FORMULAS



crest is at $x=0$

$\eta(x) = \underline{A}\cos(kx)$ A, k to be determined

at $x=0$ $\eta(0) = \underline{A = a}$

at $x=L$ $\eta(L) = a$ → $\cancel{a}\cos(kL) = \cancel{a}$

$\cos(\underbrace{kL}) = 1$ → $2\pi = kL$ →

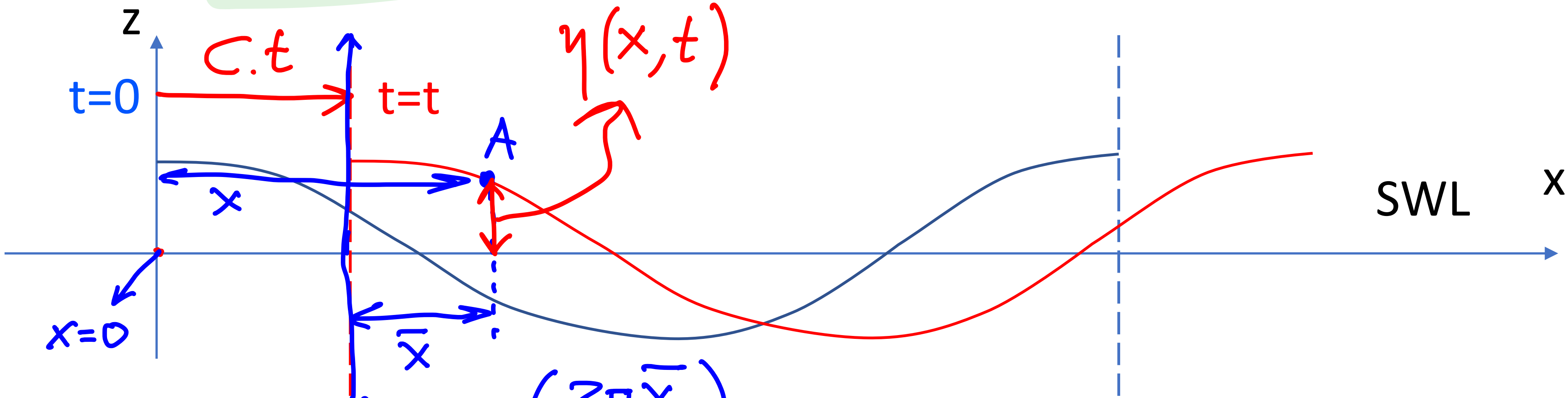
→ $\boxed{k = \frac{2\pi}{L}}$ wave number (units m^{-1} or ft^{-1})

$$\eta(x) = a\cos\left(\frac{2\pi x}{L}\right)$$

→ argument of cos has to be in radians!!

SINUSOIDAL WAVE – GOING TO THE “RIGHT”

C = wave speed or celerity



$$\eta_A(x,t) = \eta\left(\frac{2\pi \bar{x}}{L}\right)$$

$$\bar{x} = x - c.t$$

$$= a\cos\left(\frac{2\pi\bar{x}}{L}\right) = a\cos\left(\frac{2\pi}{L}(x - ct)\right) =$$

$$= a\cos\left(\frac{2\pi x}{L} - \frac{2\pi c t}{L}\right)$$

T: wave period = time it takes for a crest to propagate by a distance L

$$\boxed{L = CT} \rightarrow \boxed{C = \frac{L}{T}}$$

$\eta(x,t) = a\cos\left(\frac{2\pi x}{L} - \frac{2\pi \cancel{c} t}{\cancel{c}\cdot T}\right) \Rightarrow$

$$\boxed{\eta(x,t) = a\cos\left(\frac{2\pi x}{L} - \frac{2\pi t}{T}\right)}$$

$$\boxed{k = \frac{2\pi}{L}}$$

wave (angular) frequency : $\boxed{\omega = \frac{2\pi}{T}}$ (units rad/sec)

temporal frequency : $f = \frac{1}{T}$ (units are cycles/sec or Hertz)

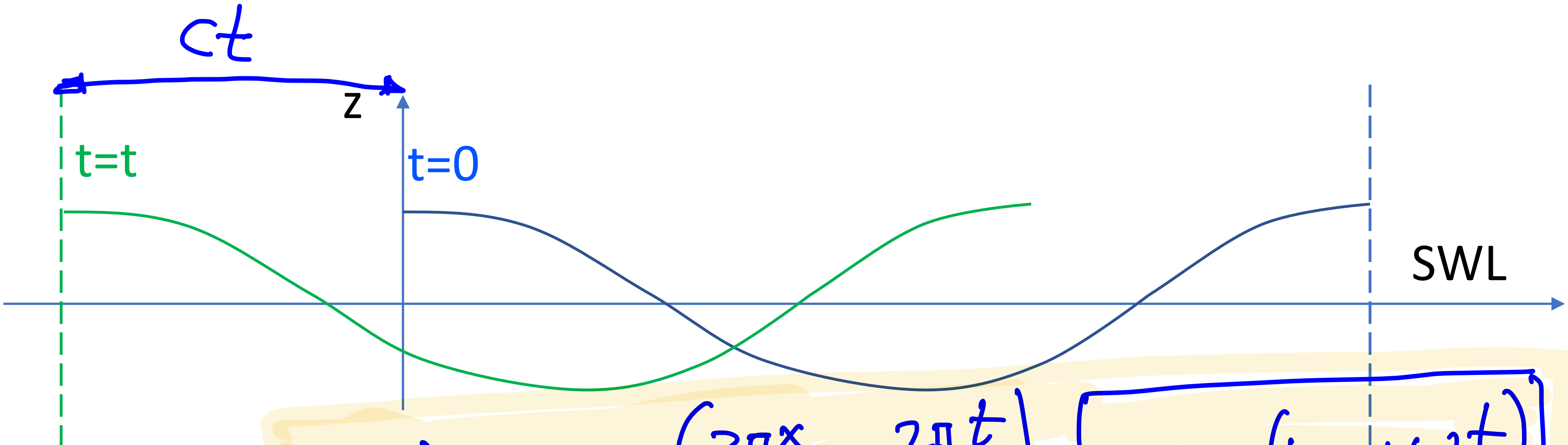
From above $\boxed{\omega = 2\pi f}$

$$\boxed{\eta(x,t) = a\cos(kx - \omega t)} \rightarrow$$

wave going to the "right" or going to $\boxed{x+}$

must be in radians!!

SINUSOIDAL WAVE – GOING TO THE "LEFT"



$$\eta(x,t) = a\cos\left(\frac{2\pi x}{L} + \frac{2\pi t}{T}\right) = \boxed{a\cos(kx+\omega t)}$$

wave going to the "left" or $-x$

H, a, L, C, T, ω, k are all positive numbers

$\eta(x,t)$ can be positive or negative!

must be in radians!!

H L T C

EXAMPLE PROBLEMS

Find the height, the length, the period, the celerity (=wave speed) of this wave, and in which direction it goes, where η and x are in meters, and t in seconds

$$\eta(x,t) = \cos(x - t)$$

1) $a = 1m \Rightarrow H = 2a = \boxed{2m}$

2) $k = 1 \rightsquigarrow \frac{2\pi}{L} = 1 \rightsquigarrow \boxed{L = 2\pi\,(m)}$

3) $\omega = 1 \rightsquigarrow \frac{2\pi}{T} = 1 \rightsquigarrow \boxed{T = 2\pi\,(sec)}$

4) $C = \frac{L}{T} = \frac{2\pi}{2\pi} = 1\ m/s$

5) goes to the "right" or $+x$

$C = \frac{\omega}{k} = \frac{\cancel{2\pi}/T}{\cancel{2\pi}/L} = \frac{L}{T}$

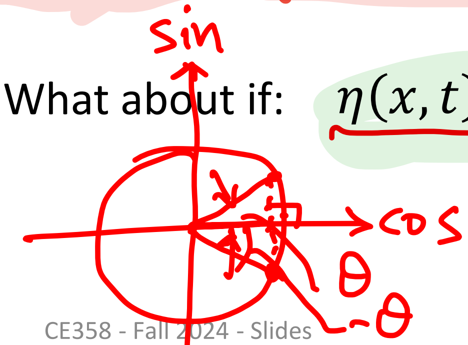
alternative formula for C

Do the same if: $\eta(x,t) = \cos(x + t)$

ALL the same (H, L, T, C), except direction

This wave goes to the "left" or $-x$

What about if: $\eta(x,t) = \cos(t - x) = \cos(x - t)$ → same as 1st problem

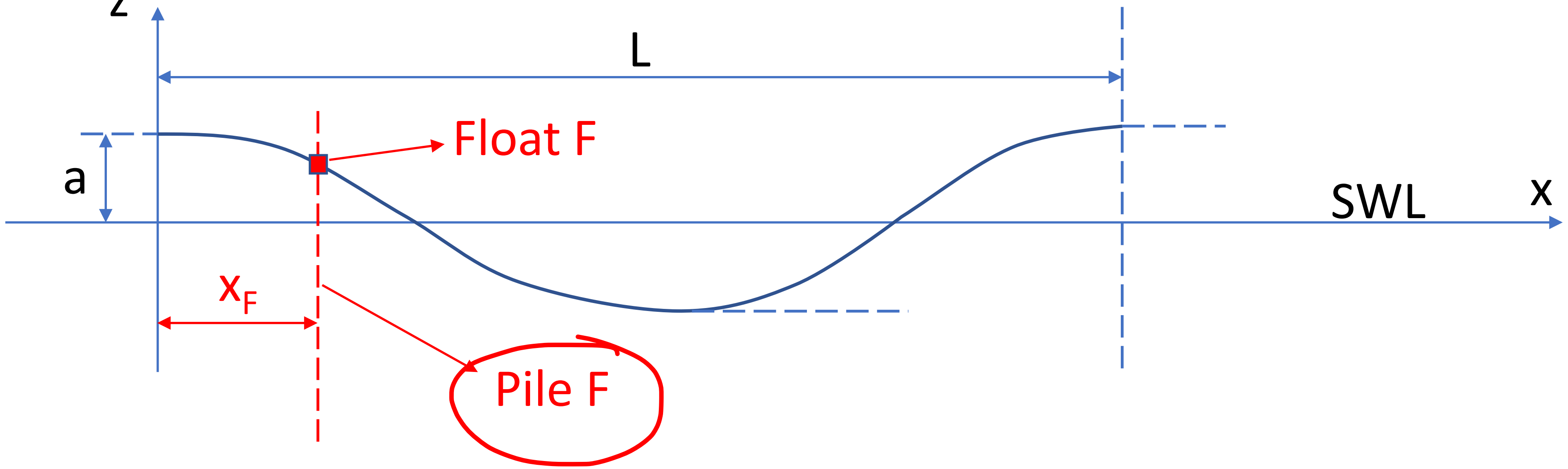


$\cos(\theta) = \cos(-\theta)$

$\sin(-\theta) = -\sin(\theta)$

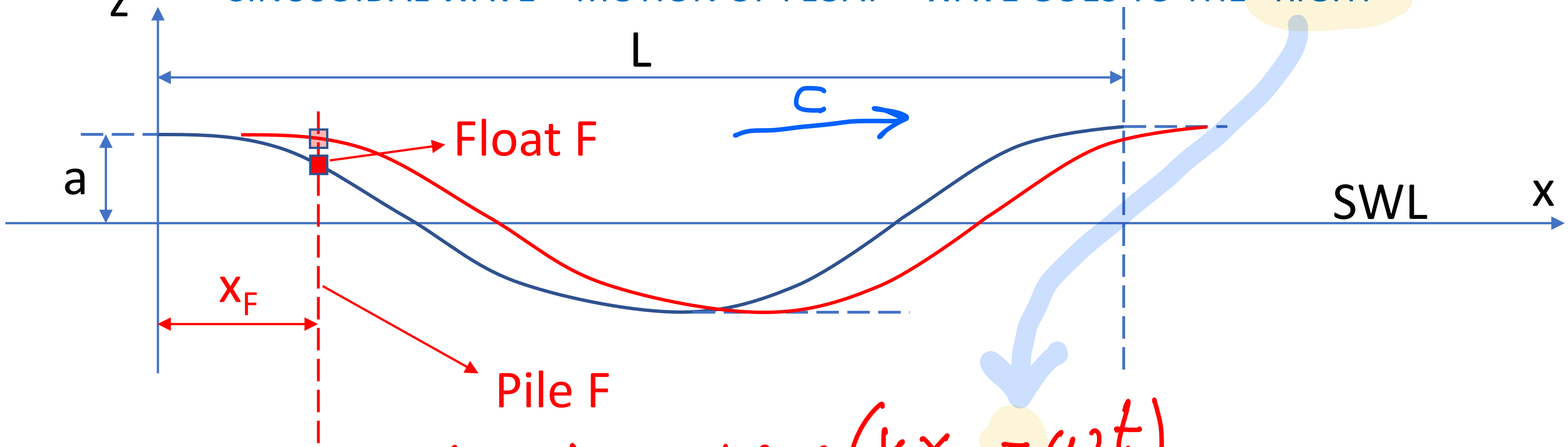
From trig.

SINUSOIDAL WAVE – CONCEPT OF FLOAT



Float F can follow the wave surface at $x = x_F$ without friction

SINUSOIDAL WAVE – MOTION OF FLOAT – WAVE GOES TO THE "RIGHT"

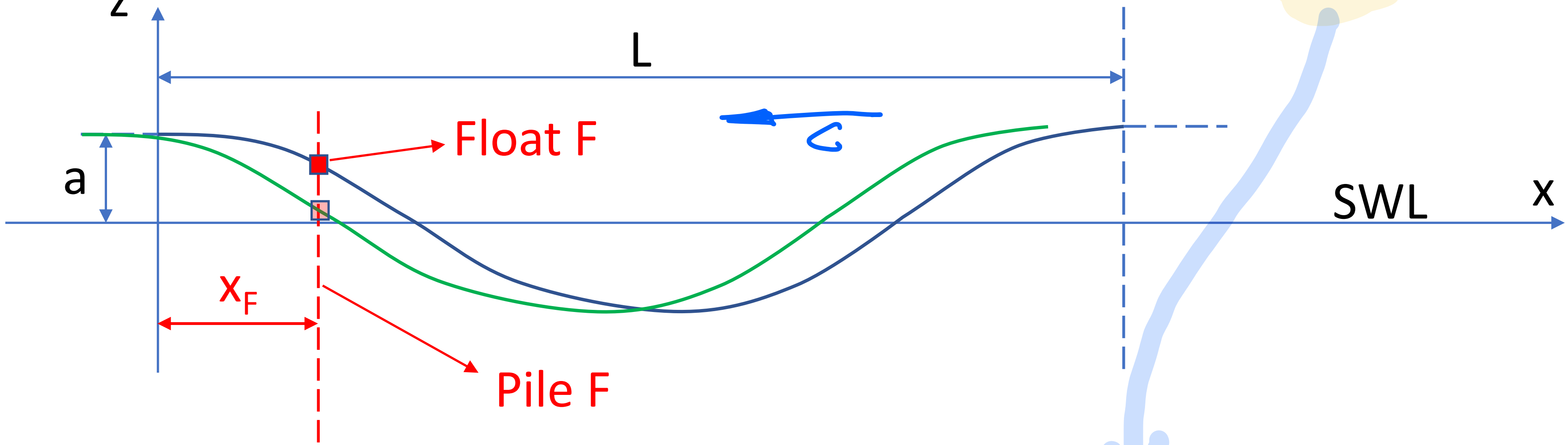


$$\eta_F(t) = \eta(x_F, t) = a\cos(kx_F - \omega t)$$

x_F is fixed

Float displacement or deflection is only a function of time

SINUSOIDAL WAVE – MOTION OF FLOAT – WAVE GOING TO THE "LEFT"



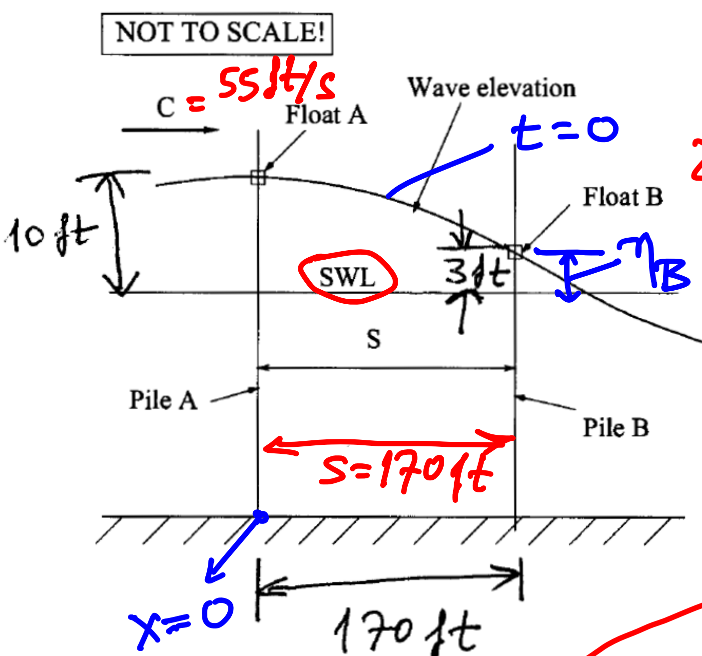
$$\eta_F(t) = \eta(x_F, t) = a\cos(kx_F + \omega t)$$

EXAMPLE PROBLEM ON FLOATS

A wave is traveling from pile A to B with a speed $C = 55\,ft/sec$ (floats A and B move freely along the piles). The distance between piles A and B is $S = 170$ ft.

At a particular time we know that float A is at its maximum level (with respect to the Still Water Line, SWL, level), equal to 10 ft. At the same instance float B is 3 ft **over** the SWL level.

2a) Find the height H, of the wave (5 points)
2b) Find the maximum wave length, L (NOTE: There is a multiplicity of solutions for L from which only the maximum is requested) (25 points)
2c) The period of the wave, T (5 points)
2d) How long after A and B will have the same elevation? What is the value of this elevation? (25 points)



2a) $H = ?$

$a = 10\,ft \Big\}\; H = 2a = 20\,ft$

2b) $\eta(x,t) = a\cos(kx - \cancel{\omega t})$

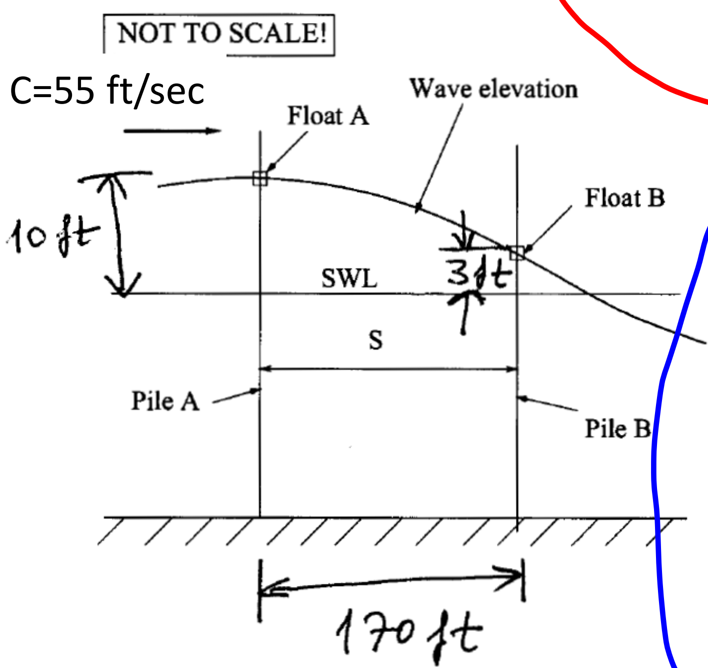
$\eta_B = 3\,ft$

$\eta_B = a\cos(k x_B)$ $(t = 0)$

$3\,ft = 10\,ft\cos(k s)$ $S = 170\,ft$

EXAMPLE PROBLEM ON FLOATS

a) H=? b) L=? c) T=? d) t=?



$$\boxed{0.3 = \cos(ks)}$$

If: $\cos(\alpha) = \cos(\beta) \Rightarrow$ (from trig.)

$\Rightarrow \begin{cases} \alpha = \pm\beta + n2\pi \\ n = 0, \pm1, \pm2, \ldots \end{cases}$ general solution

$\arccos(0.3) = \cos^{-1}(0.3) = 1.266\,\text{rad}$

$\cos(1.266) = \cos(ks)$ or $\cos(ks) = \cos(1.266)$

$s = 170\,ft$, $k = \frac{2\pi}{L}$

$ks = \pm 1.266 + n2\pi$; $n = 0, \pm1, \pm2, \ldots$

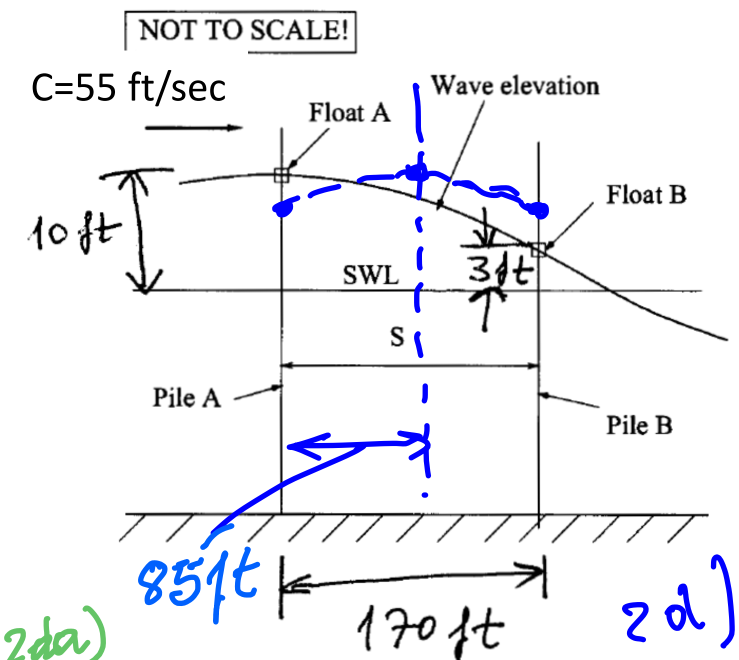
a) $ks = 1.266 + n2\pi$

$L > 0$ and L max which means k has to be min

$n = 0 \Rightarrow \boxed{L = 843.7\,ft}$ ✓

EXAMPLE PROBLEM ON FLOATS

a) H=? b) L=? c) T=? d) t=?



b) $ks = -1.266 + n2\pi \quad (n = 0, \pm 1, \pm 2, ..)$

$n = 1$ for $k > 0$ and k min

$\Rightarrow \boxed{L = 212.9\,ft}$

2c) $\boxed{T} = \frac{L}{c} = \frac{843.7}{55} = \boxed{15.34\,sec}$

2d) crest must be in the middle between A & B so that $\eta_A = \eta_B$

2da) Intuitive approach $\eta_A = \eta_B$ when crest in the middle between A&B

$\boxed{t =} \frac{85\,ft}{55\,ft/s} = \boxed{1.545\,sec}$

2db) Using trig: $\eta_A = \eta_B$

$\cancel{a}\cos(k\underbrace{\cancel{x_A}}_{=0} - \omega t) = \cancel{a}\cos(k x_B - \omega t)$

$= S = 170\,ft$

$$\cos(-\omega t) = \cos(ks - \omega t) \Rightarrow t = \ldots$$

$$\cos(\omega t) = \cos(ks - \omega t)$$

$$\omega t = \pm(ks - \omega t) + 2\pi n \qquad n = 0, \pm 1, \pm 2, \ldots$$

a) $\omega t = ks - \omega t + 2\pi n \Rightarrow \underline{2\omega t = ks + 2\pi n}$

$t > 0$ $\quad$ $t = \min$ (for the first time)

$n = 0$ $\longrightarrow$ $2\omega t = ks \longrightarrow t = \dfrac{ks}{2\omega} = 1.545\ \text{sec}$

the same answer as from the intuitive method

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{15.34} = 0.4096\ (\text{rad/sec})$$

$$k = \frac{2\pi}{L} = \frac{2\pi}{843.7} = 0.00745\ \text{ft}^{-1}$$

$S = 170\ \text{ft}$

b) $\omega t = -(ks - \omega t) + 2\pi n$

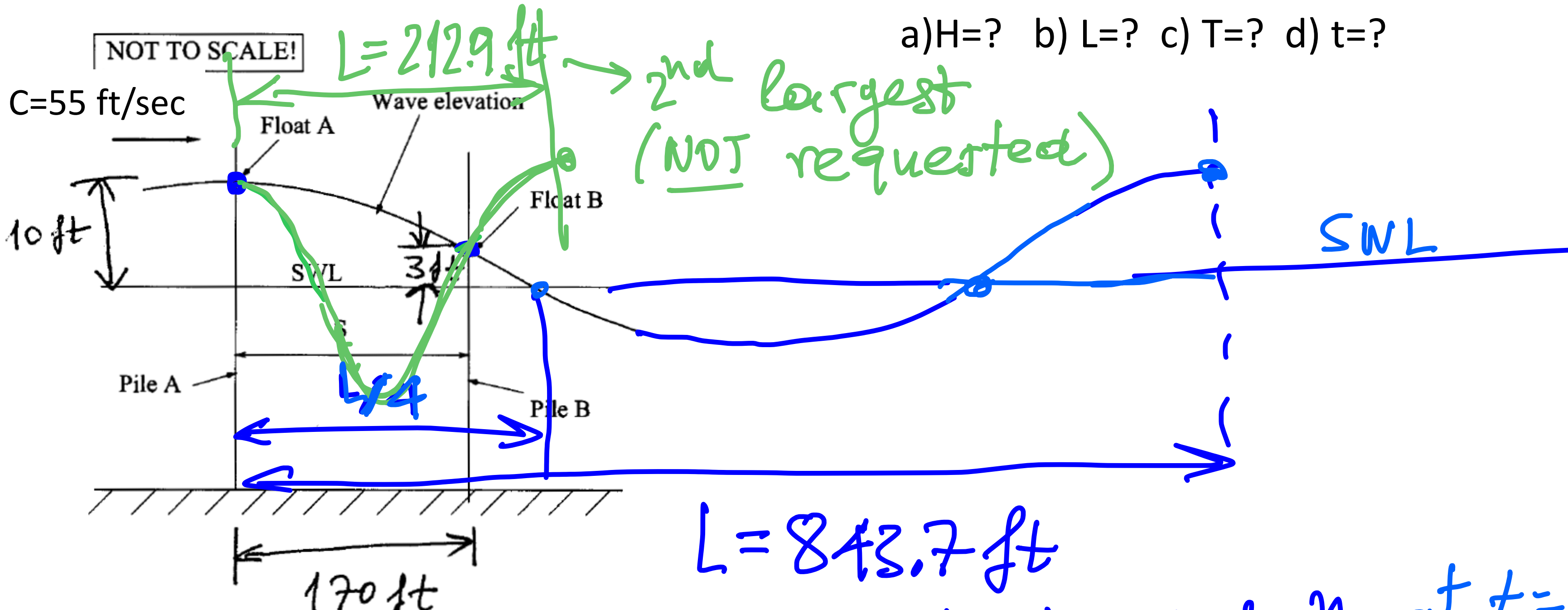
$$\cancel{\omega t} = -ks + \cancel{\omega t} + 2\pi n$$

not useful since it does not provide a formula for t

EXAMPLE PROBLEM ON FLOATS

a) H=? b) L=? c) T=? d) t=?

C=55 ft/sec



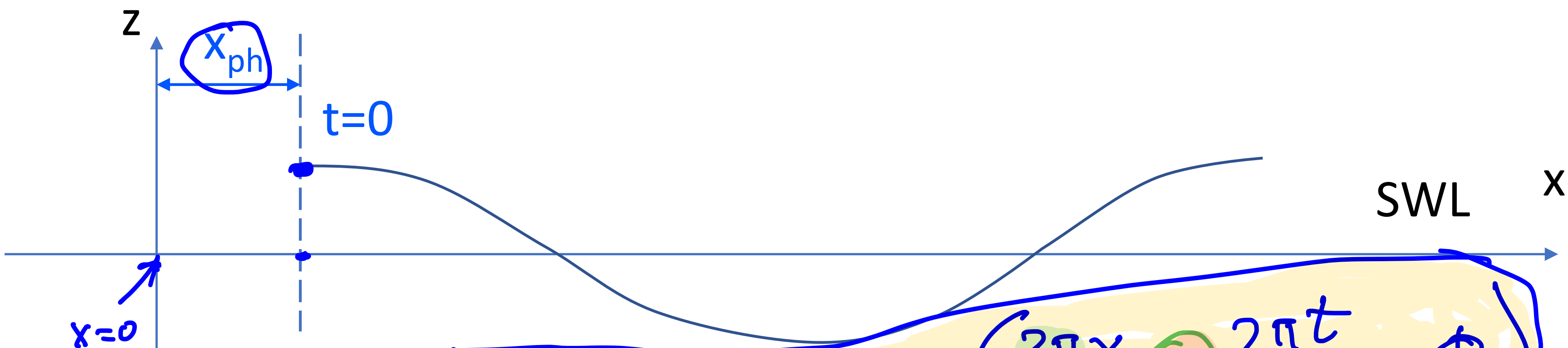
We are also requested to determine η_A & η_B at $t=1.545$ sec

$$\eta_A = a\cos(k\underbrace{\cancel{x_A}}_{=0}-\omega t) = a\cos(\omega t) = 10\cos(0.4096\times 1.545) = 8.06\ ft$$

$$\eta_B = a\cos(kx_B-\omega t) = 10\cos(0.00745\times 170 - 0.4096\times 1.545) = 8.06\ ft$$

Note $\eta_A=\eta_B$ (except for small roundoff errors) and that checks our previous result for t

SINUSOIDAL WAVE – PHASE OF A WAVE



$$\eta(x,t) = a\cos\left(\frac{2\pi x}{L} \pm \frac{2\pi t}{T} - \phi\right)$$

"Standard" or "canonical" form

ϕ: phase of the wave

$$\boxed{\phi = \frac{x_{ph}}{L}\, 2\pi}$$

(+) wave goes to (−x)
(−) " " " (+x)

the factor multiplying x has to be positive

Note ϕ is not unique since we can add or subtract multiples of 2π without changing η!

EXAMPLES ON PHASE OF A WAVE

Put the following wave profiles into their "canonical" form and determine their phase and their direction of propagation. Plot the wave profiles at $t = 0$ and verify that the phases you determined *make sense*. Remember a, H, L, k, T, ω, and C, are, by definition, **positive** numbers.

(a) $\eta = \sin(x - 2t)$

(b) $\eta = -\cos(3x + t)$

$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$

(a) $\eta = \sin(x - 2t) = \cos\left(\frac{\pi}{2} - (x - 2t)\right) =$

$= \cos\left(\frac{\pi}{2} - x + 2t\right) = \boxed{\cos\left(x - 2t - \frac{\pi}{2}\right)}$

$\cos(\theta) = \cos(-\theta)$

$\phi = \frac{\pi}{2}$

to the "right" or $+x$

(b) $= \eta = -\cos(3x + t) = \cos\left(\pi - (3x + t)\right)$

$-\cos\theta = \cos(\pi - \theta)$

$\cos(\pi - 3x - t) = \boxed{\cos(3x + t - \pi)}$

$\cos(-\theta) = \cos(\theta)$

$\phi = \pi$

to the left or $-x$

EXAMPLES ON PHASE OF A WAVE

Put the following wave profiles into their "canonical" form and determine their phase and their direction of propagation. Plot the wave profiles at $t = 0$ and verify that the phases you determined *make sense*. Remember a, H, L, k, T, ω, and C, are, by definition, **positive** numbers.

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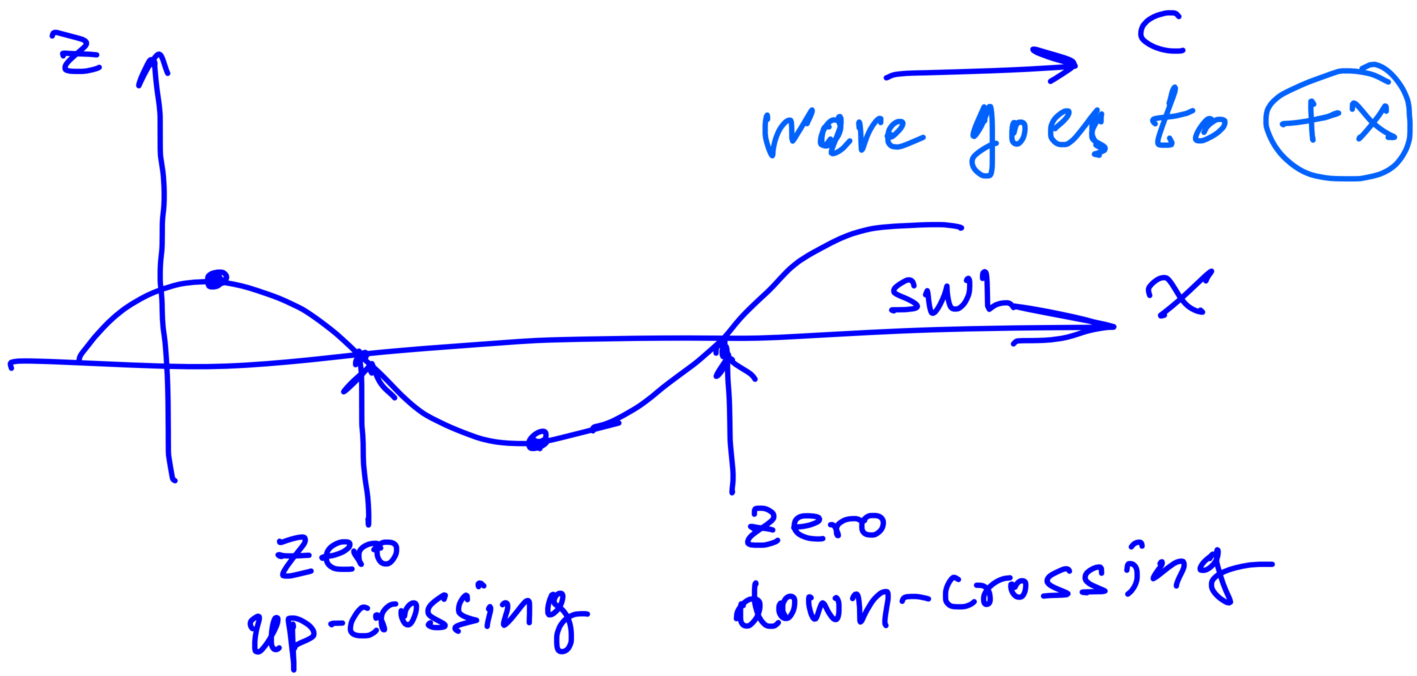
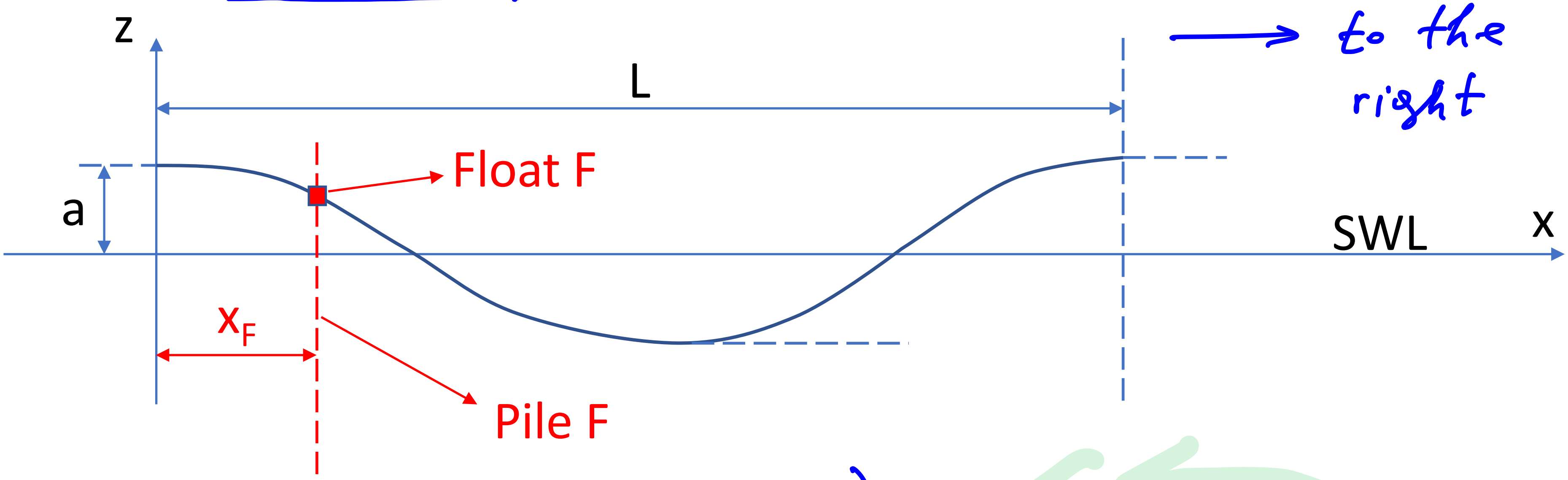


DISPLACEMENT, VELOCITY, AND ACCELERATION OF A FLOAT



$\rightarrow$ to the right

$$\eta_F = a\cos(kx_F - \omega t)$$

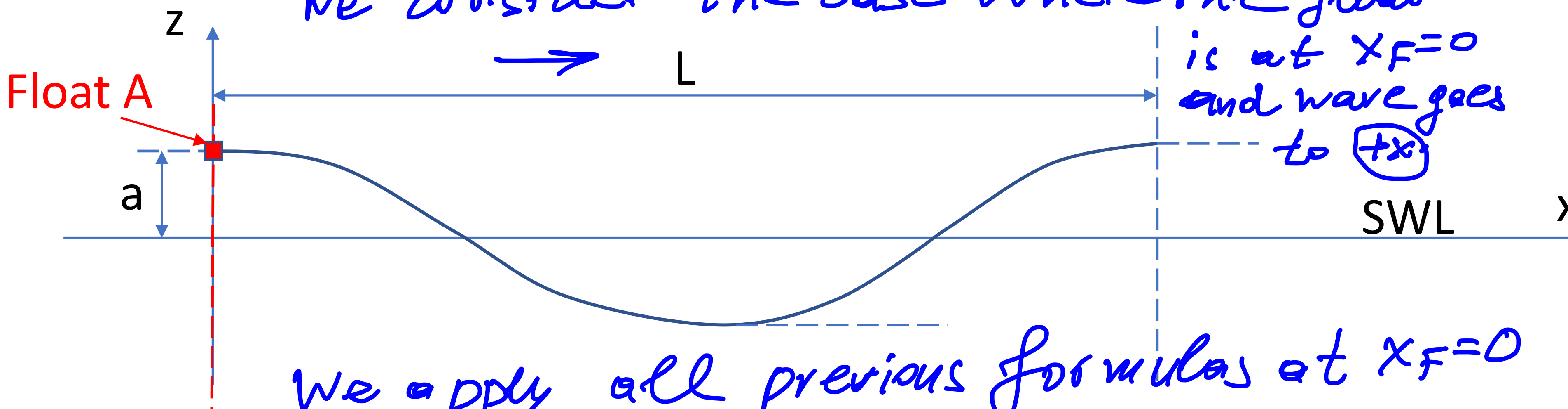
$$V_F = \frac{\partial \eta}{\partial t} = a\left[-\sin(kx_F - \omega t)\right](-\omega) = a\omega\sin(kx_F - \omega t)$$

$$a_F = \frac{\partial V_F}{\partial t} = a\omega\cos(kx_F - \omega t)(-\omega) = -a\omega^2\cos(kx_F - \omega t)$$

$$\Rightarrow \underline{a_F = -\omega^2 \eta_F}$$

EXAMPLE ON DISPLACEMENT, VELOCITY, AND ACCELERATION OF A FLOAT

We consider the case where the float is at $x_F = 0$ and wave goes to $+x$

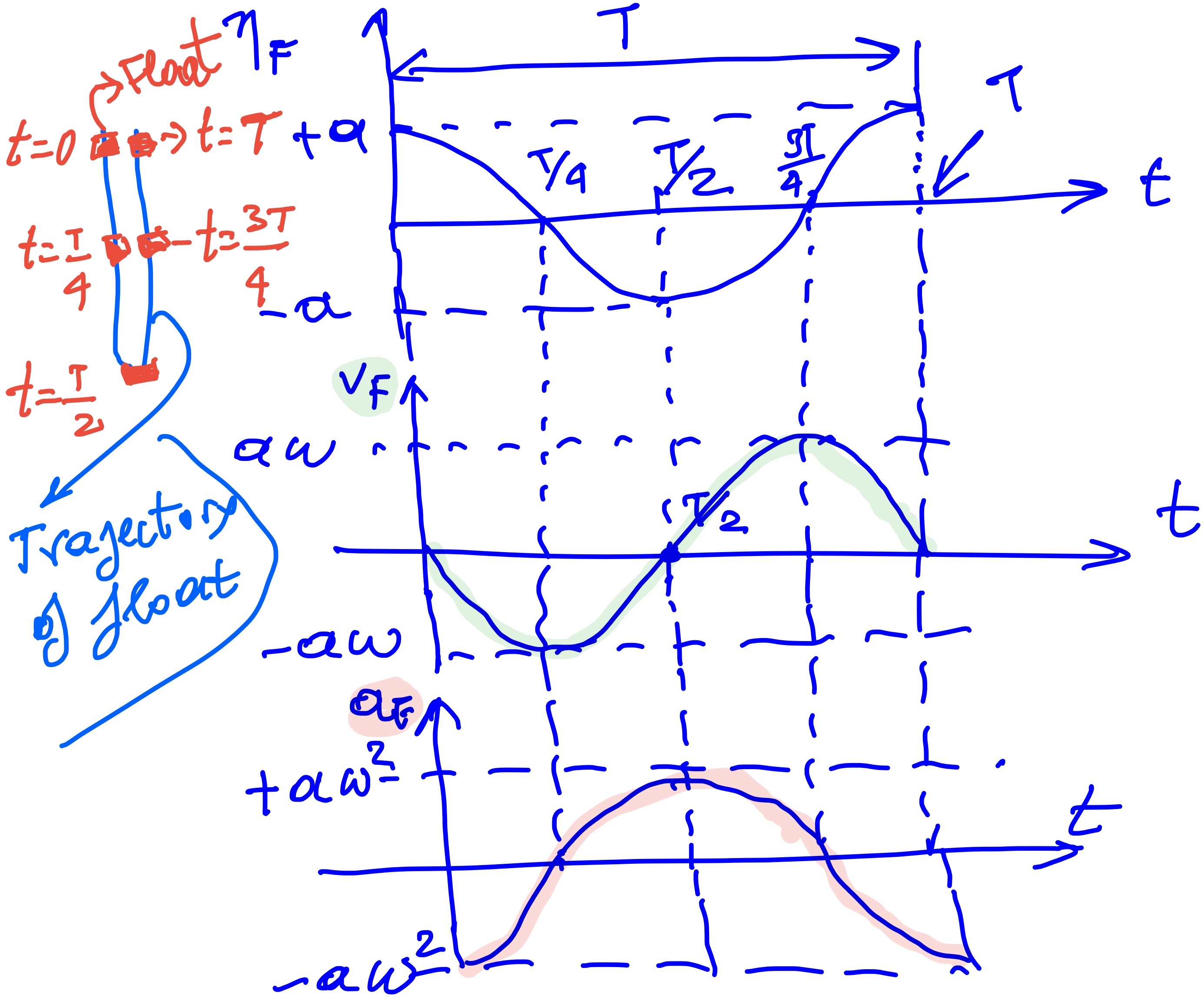


We apply all previous formulas at $x_F = 0$

$$\eta_F = a\cos(\omega t)$$

$$V_F = a\omega\sin(-\omega t) = -a\omega\sin(\omega t)$$

$$a_F = -a\omega^2\cos(\omega t)$$



$$\eta_F = a\cos(\omega t)$$

$$V_F = -a\omega\sin(\omega t)$$

$$a_F = -a\omega^2\cos(\omega t)$$

Motion of float is analogous to motion of a person on a swing: