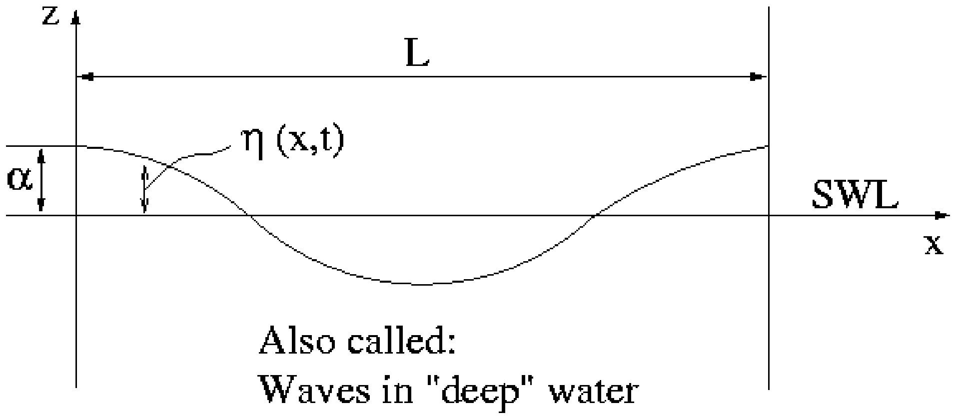
# LINEAR WAVE THEORY – DEEP WATER




**Finally :**

$$\varphi(x,z,t) = \frac{a \cdot \omega}{k} \cdot e^{kz} \cdot \sin(kx - \omega t) \qquad (12)$$

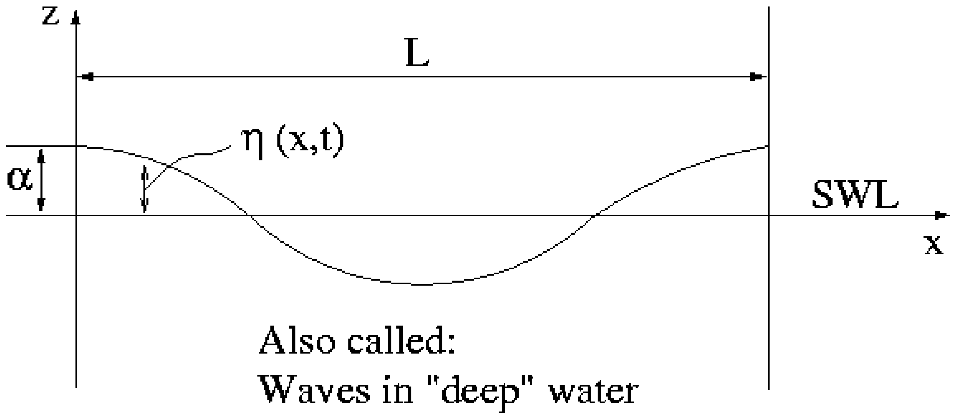
$$u(x,z,t) = \frac{\partial \varphi}{\partial x} = a\omega e^{kz} \cos(kx - \omega t) \qquad (28)$$

$$w(x,z,t) = \frac{\partial \varphi}{\partial z} = a\omega e^{kz} \sin(kx - \omega t) \qquad (29)$$

$$a_x = a\omega^2 e^{kz} \sin(kx - \omega t) \qquad (58)$$

$$a_z = -a\omega^2 e^{kz} \cos(kx - \omega t) \qquad (59)$$

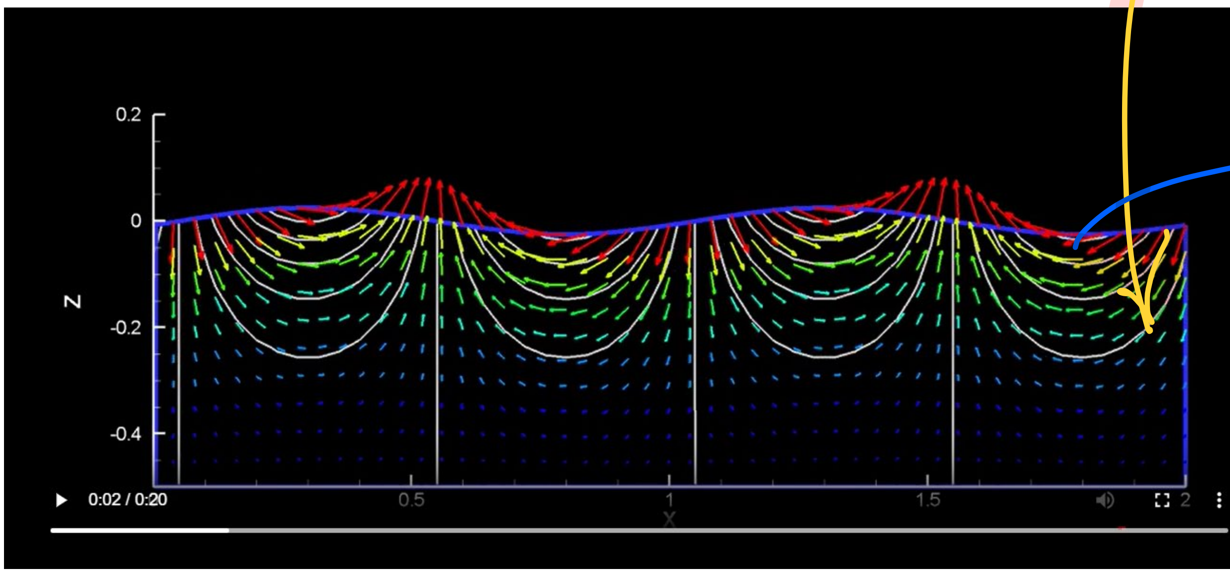
# LINEAR WAVE THEORY - DEEP WATER - PARTICLE TRAJECTORIES




The pathlines (or particle trajectories) for deep $H_2O$ were addressed in the previous set of slides.

streamlines $\neq$ pathlines

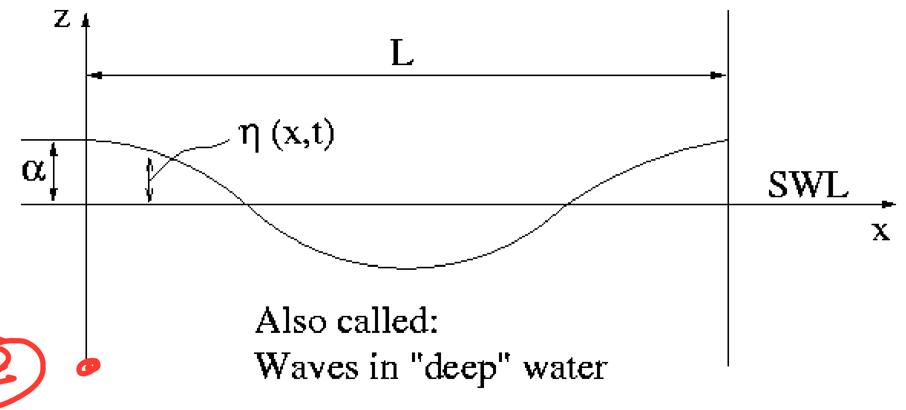
Since this is UNSTEADY FLOW




*How about the streamlines of the flow-field under the wave?*

# LINEAR WAVE THEORY – DEEP WATER - PRESSURES

$$\varphi(x,z,t) = \frac{a \cdot \omega}{k} \cdot e^{kz} \cdot \sin(kx - \omega t) \qquad (12)$$




Bernoulli:

$$\rho \frac{\partial \phi}{\partial t} + p + \frac{\rho q^2}{2} + \rho g z = 0$$

We make the constant zero. An explanatory is provided on the class website!

$$\rightarrow p = -\rho \frac{\partial \phi}{\partial t} - \cancel{\frac{\rho q^2}{2}} - \rho g z$$

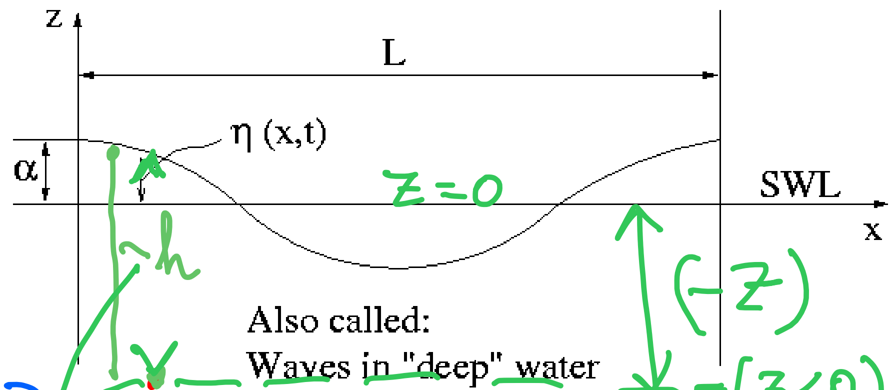
H.O.T.

$\sim a^2$

$$\frac{\partial \phi}{\partial t} = \frac{a\omega}{k} e^{kz} \cos(kx - \omega t)(-\omega)$$

# LINEAR WAVE THEORY – DEEP WATER - PRESSURES

$$\varphi(x,z,t) = \frac{a\cdot\omega}{k}\cdot e^{kz}\cdot\sin(kx-\omega t) \qquad (12)$$




$$\boxed{p} = -\rho\left[\frac{a\omega}{k}e^{kz}\cos(kx-\omega t)(-\omega)\right] - \rho g z$$

$\eta$

$$= \boxed{\rho g \eta e^{kz} - \rho g z}$$

$\frac{\omega^2}{k} = g$ (due to dispersion relationship)

$p = \rho g h = \rho g(\eta - z)$

$h = \eta - z$

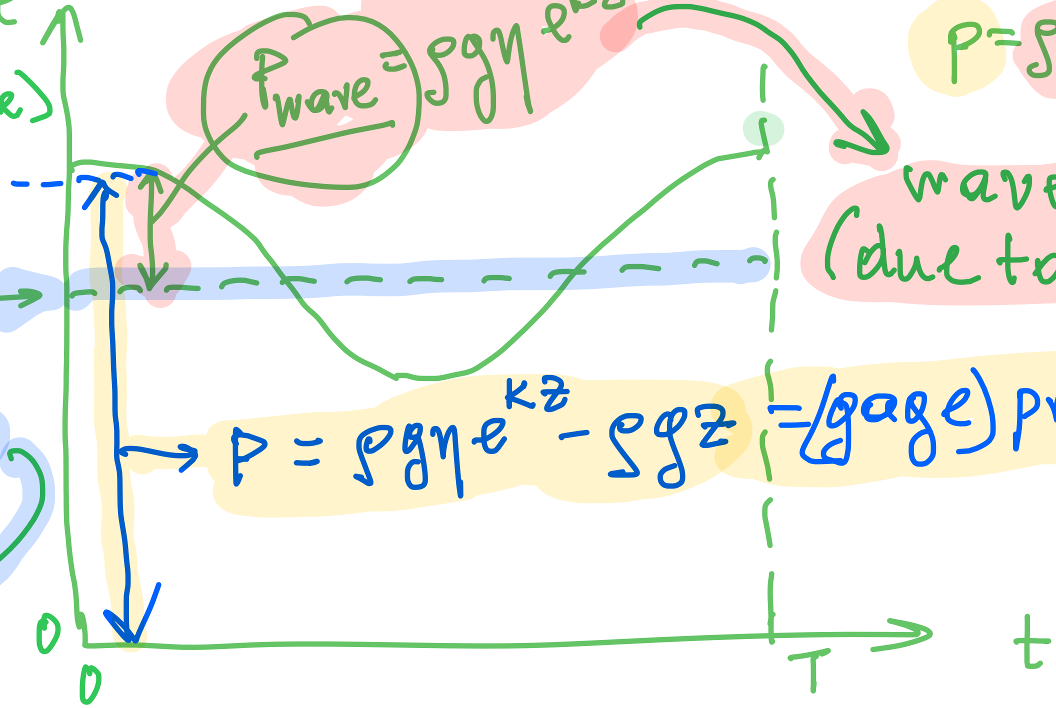
$\cancel{\rho g \eta - \rho g z}$

On the free surface: $p = \rho g \eta \underbrace{e^{k\eta}}_{\approx 1} - \rho g \eta = 0$ (good check)

It includes effects of unsteadiness through $\rho\,\partial\phi/\partial t$ term in Bernoulli

WRONG!!

It does not include effects of unsteadiness (it is quasi-steady) pressure $p$ (gage)




mean pressure $= -\rho g z$ $(z<0)$

Also the pressure in the absence of waves

$\underline{p_{wave}} = \rho g \eta e^{kz}$

$p = \rho g \eta e^{kz} \underline{- \rho g z}$

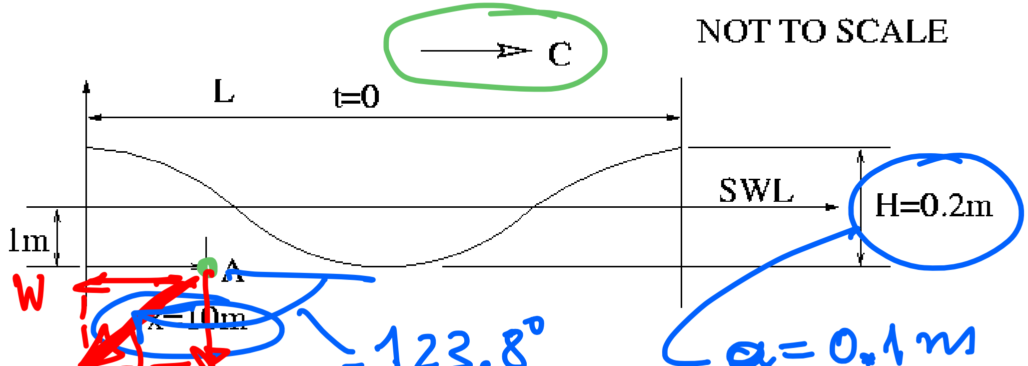
$=$ wave pressure (due to waves ONLY)

$p = \rho g \eta e^{kz} - \rho g z$ = (gage) pressure

# LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 1

5sec sinusoidal wave is propagating in deep water. Find the velocity vector and wave pressure at a distance = 10m from the crest, depth of 1m and the time t=3 sec.

$= T$ $\quad \vec{q}$ $\quad P_{wave}$




$$u = a\omega e^{kz}\cos(kx-\omega t)$$

$$w = a\omega e^{kz}\sin(kx-\omega t)$$

$a = 0.1\,m$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{5} = 1.257 \text{ rad/sec}$$

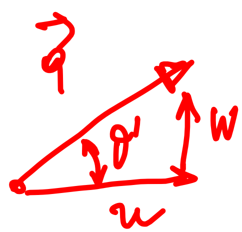
$$e^{kz} = e^{(0.161)(-1)} = 0.85$$

$$L = \frac{gT^2}{2\pi} = \frac{9.81\times 5^2}{2\pi} = 39m$$

$$k = \frac{2\pi}{L} = \frac{2\pi}{39} = 0.161\,m^{-1}$$

$$\theta = kx-\omega t = 0.161\times 10m - 1.257\times 3 = -2.161 \text{ rad}$$

$u = -0.06\ m/s$ ; $w = -0.089\ \frac{m}{s}$

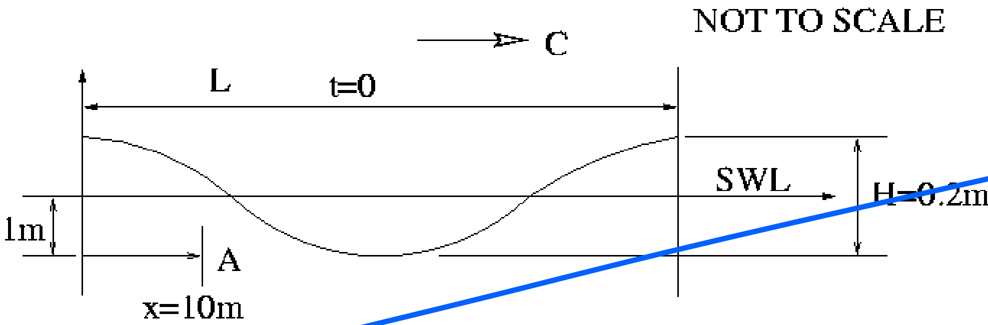



$$\tan\theta' = \frac{w}{u} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

in our case $\theta = -2.161 \text{ rad} = -123.8°$

$$\theta' = \theta$$

5sec sinusoidal wave is propagating in deep water. Find the velocity vector and wave pressure at a distance = 10m from the crest, depth of 1m and the time t=3 sec.




$$P_{wave} = \rho g \eta e^{kz}$$

$$g = 9.81 \ \frac{m}{s^2} \qquad e^{kz} = 0.85$$

$$\rho = 1{,}000 \ \frac{kg}{m^3} \quad \text{fresh-water}$$

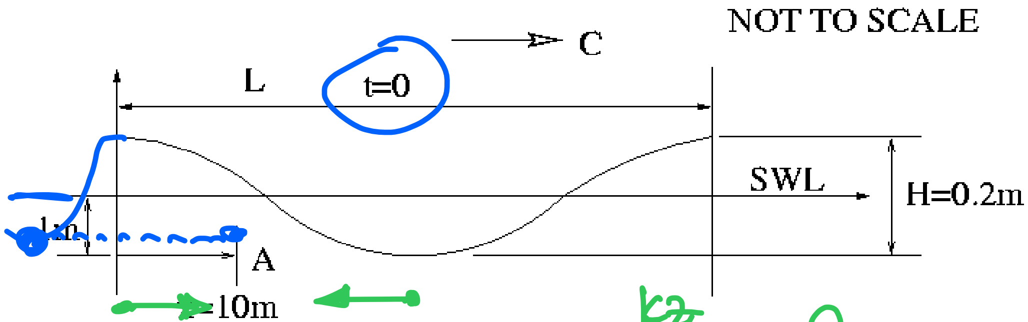
$$\rho_{sea} = 1{,}025 \ \frac{kg}{m^3} \quad \text{sea-water}$$

$$\eta = a \cos(kx - \omega t) = 0.1 \cos(-2.161 \text{ rad}) = -0.05565 \text{ m}$$

$$\boxed{P_{wave}} = 1{,}025 \times 9.81 \times (-0.05565 \text{ m}) \times (0.85) = \boxed{-457.6 \text{ Pa}}$$

# LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 1

5sec sinusoidal wave is propagating in deep water. Find the velocity vector and wave pressure at a distance = 10m from the crest, depth of 1m and the time t=3 sec.




What would your answer be if we asked for the moment (t) when a trough is above A

$u = a\omega e^{kz}\cos\theta = -a\omega e^{kz} = \boxed{0.107\ \frac{m}{s}}$

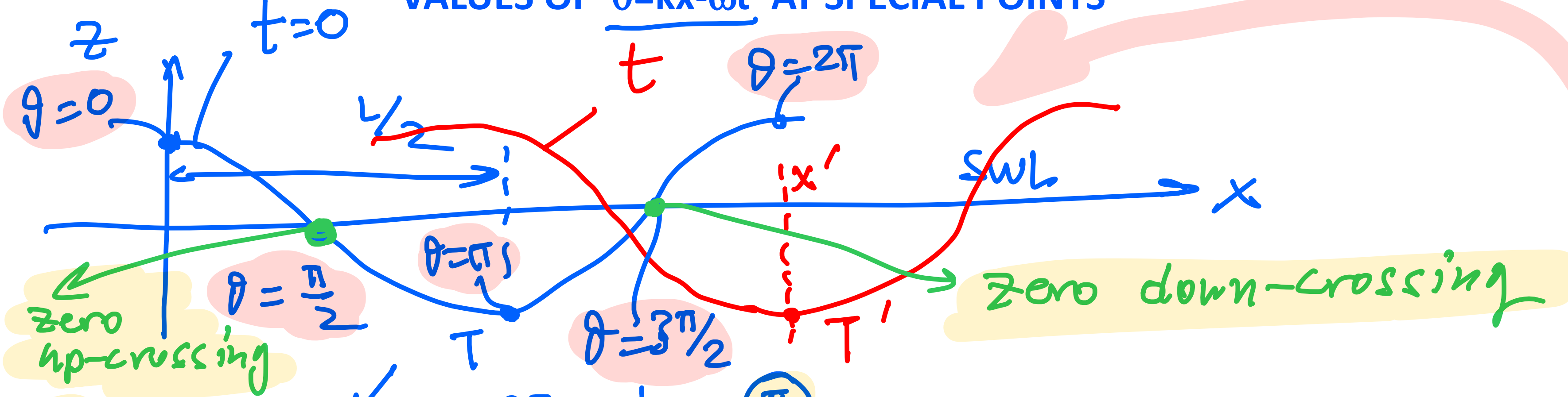
$w = a\omega e^{kz}\sin\theta = 0$

$\vartheta = \pi$

$P_{wave} = \rho g \eta e^{kz} = 1{,}025 \times 9.81 \times (-0.1) \times (0.85) = \boxed{-854.7\ Pa}$

see next slide which explains that $\theta = kx - \omega t = \pi$ ALWAYS at a trough

# VALUES OF $\theta = kx - \omega t$ AT SPECIAL POINTS




$$\theta_T = kx - \underbrace{\cancel{\omega t}}_{=0} = \frac{2\pi}{L}\cdot\frac{L}{2} = \pi$$

$$\theta_{T'} = kx' - \omega t = \frac{2\pi}{L}\left(\frac{L}{2} + c\cdot t\right) - \omega t =$$

$$x' = \frac{L}{2} + c\cdot t$$

$$= \pi + \cancel{\frac{2\pi}{L}\cdot ct - \omega t} = 0$$

$$T = \frac{L}{c} \qquad \frac{2\pi}{T}t = \omega\cdot t$$

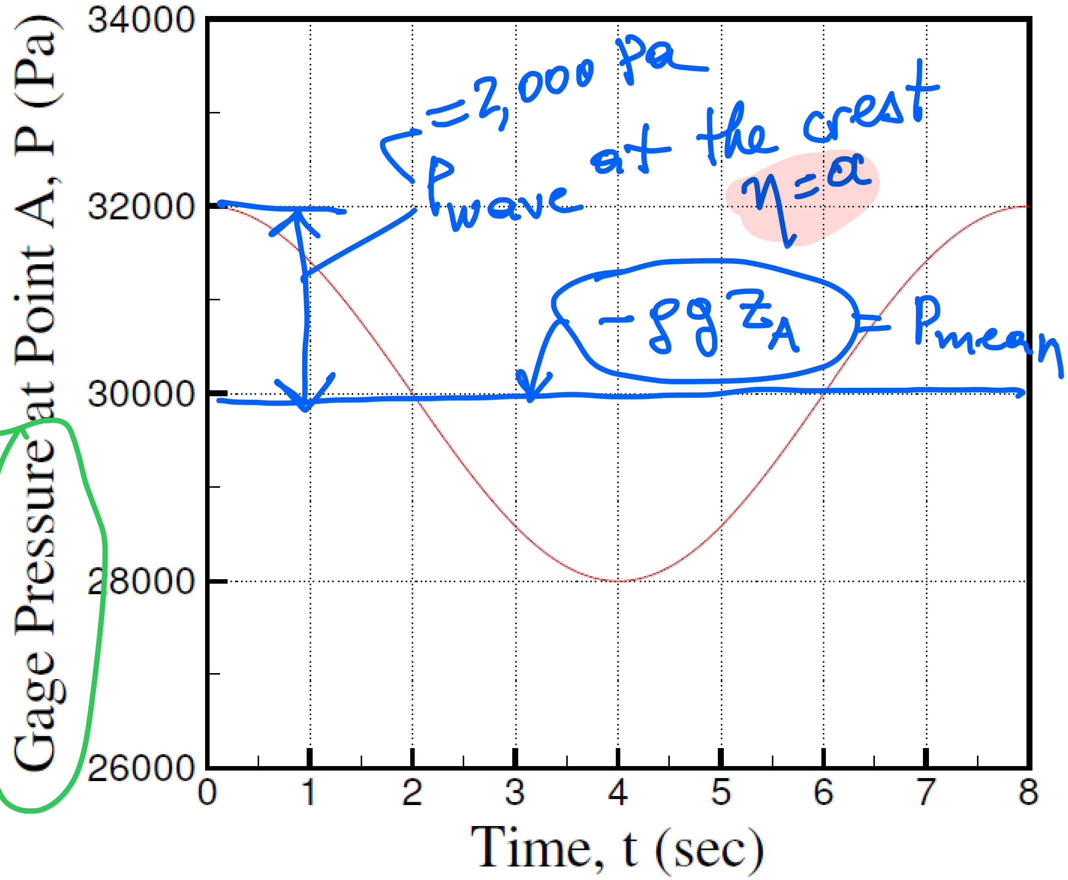
$$\Rightarrow \boxed{\theta_{T'} = \pi}$$

Thus $\theta_{T'} = \theta_T = \pi$ at a trough ALWAYS

The $\theta$ value for other special points is shown on the graph at the top

# LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 2

A sinusoidal wave is propagating in infinite depth sea-water (in the +x direction). A pressure gage is mounted at point A under the free surface. The time history of the gage pressure at point A over one wave period is shown in the figure below. Using the information on the given graph, apply linear wave theory and determine the following:

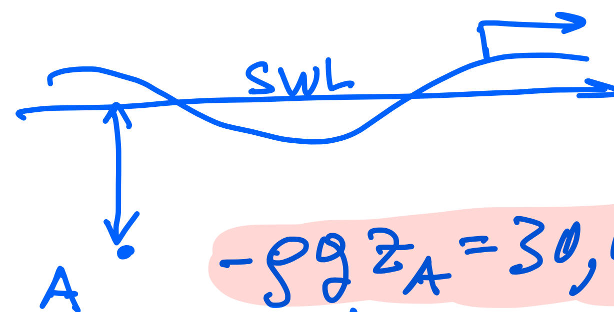



**a)** The depth at point A. (5 points)

**b)** The wave height. (10 points)

**c)** The wave elevation above point A at t = 6.8 sec. (10 points)

**d)** The values of the *horizontal* particle velocity and acceleration at point A and at t = 6.8 sec. (10 points)



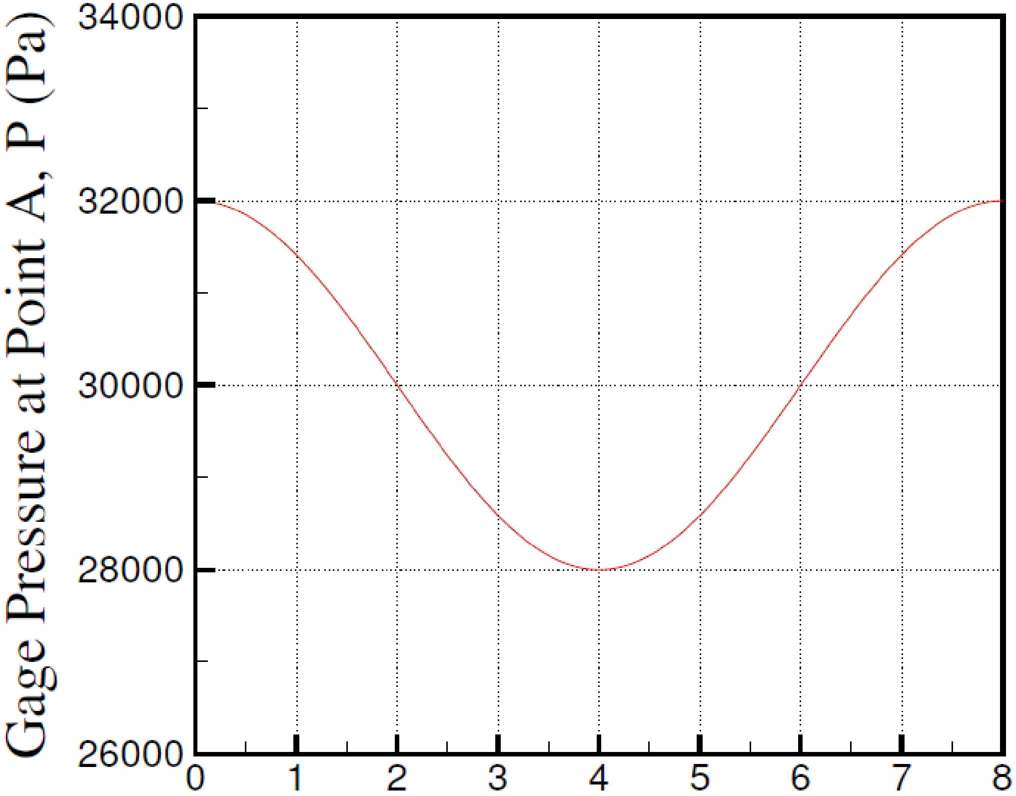

$-\rho g z_A = 30,000$

$\rho$: 1,025  $g$: 9.81

$\Rightarrow \boxed{z_A = -2.98\,m}$

so depth at point A is 2.98 m

# LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 2




**a)** The depth at point A. (5 points)

**b)** The wave height. (10 points)

b) $P_{wave} = 32{,}000 - 30{,}000$

$\rho g \eta e^{kz_A} = 2{,}000$ $\quad (z_A = -2.98m)$

$\eta = a$

$T = 8\,sec$

$L = \frac{gT^2}{2\pi} = 100m$

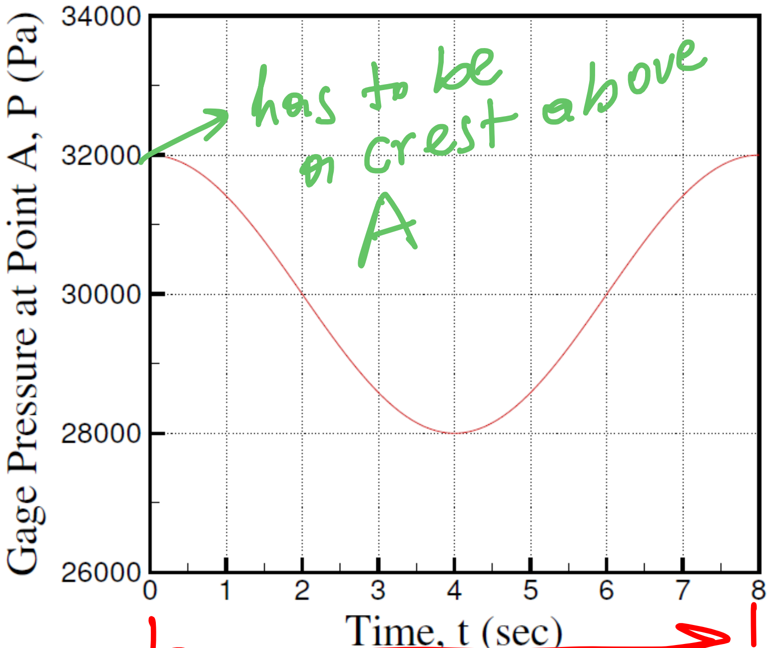
$k = \frac{2\pi}{L} = 0.0628\,m^{-1}$

$e^{kz_A} = 0.829$

$\Rightarrow a = 0.24m$

$\boxed{H} = 2a = \boxed{0.48m}$

# LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 2




has to be a crest above A

T = 8 sec

**c)** The wave elevation above point A at t = 6.8 sec. (10 pts)

**d)** The values of the *horizontal* particle velocity and acceleration at point A and at t = 6.8 sec. (10 pts)

c) $\eta = a \cos(kx - \omega t)$

$a$: 0.24 m

$k$: 0.0628 $m^{-1}$

x = 0

due to the fact that crest is at x=0 at t=0

T = 8 sec

$\omega = \frac{2\pi}{8} = 0.785 \; \frac{rad}{sec}$

$\vartheta = kx - \omega t = -0.785 \times 6.8 = -5.43 \text{ rad}$

$\eta = 0.24 \times \cos(-5.43 \text{ rad}) = 0.141 \text{ m}$

d) $u = ?$, $a_x = ?$ at t = 6.8 sec

$u = a\omega e^{kz_A} \cos(\theta) = 0.24 \times 0.785 \times 0.829 \times \cos(-5.43 \text{ rad})$

$\boxed{u = 0.092 \; \frac{m}{s}}$

$a_x = a\omega^2 e^{kz_A} \sin(\theta) = 0.24 \times (0.785)^2 \times 0.829 \times \sin(-5.43 \text{ rad})$

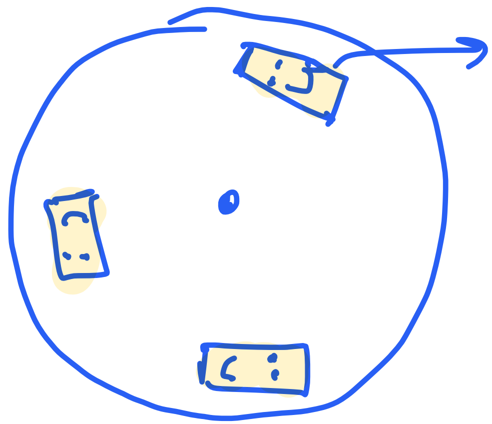
$\boxed{a_x = 0.099 \; m/s^2}$


We have assumed IRROTATIONAL FLOW or $\omega = 0$

Ferris wheel

person does not spin

Merry Go Round



person spins around themselves

Fluid particles under the wave move on circular trajectories but DO NOT spin around themselves, thus $\underbrace{\omega}_{\text{vorticity}} = 2\underbrace{\Omega}_{\text{spin}} = 0$