1 The momentum equations for viscous flows in 2-D

We consider a fluid particle in two dimensions (x, z) as shown in Figure 1, along with the normal stresses σ_{xx} and σ_{zz} and shear stresses τ acting on all its sides. The fluid is assumed to be incompressible with density ρ. The flow-field has velocities u and w, and the particle is subjected to accelerations a_x and a_z and a gravitational acceleration g as shown in Figure 1. In this section we will derive the momentum equations that govern the flow.

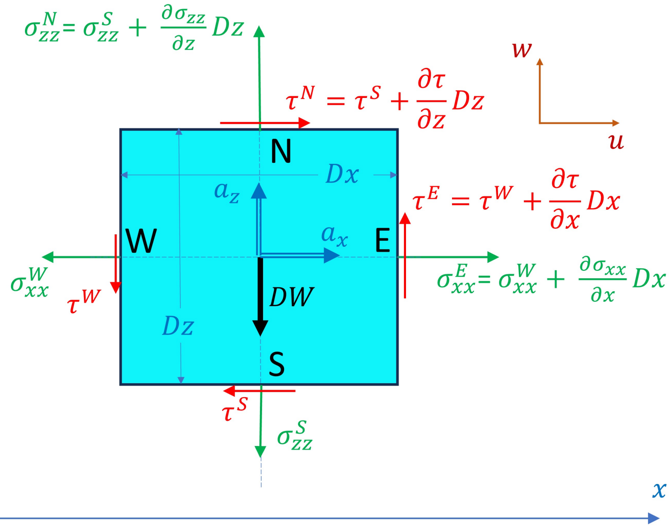

Figure 1: Fluid particle in 2-D and the stresses acting on it.

First we apply Newton's Law along x:

$$\Sigma F_x = (Dm)a_x \tag{1}$$

or

$$\sigma_{xx}^E Dz - \sigma_{xx}^W Dz + \tau^N Dx - \tau^S Dx = (Dm)a_x \tag{2}$$

or

$$(\sigma_{xx}^E - \sigma_{xx}^W)Dz + (\tau^N - \tau^S)Dx = (Dm)a_x \tag{3}$$

or

$$\frac{\partial \sigma_{xx}}{\partial x} DxDz + \frac{\partial \tau}{\partial z} DzDx = (Dm)a_x \tag{4}$$

but

$$Dm = \rho DxDz \tag{5}$$

and

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \tag{6}$$

Finally equation (4) leads to the momentum equation along x:

$$\boxed{\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right]=\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau}{\partial z}} \tag{7}$$

Similarly, by applying Newton's Law along z:

$$\Sigma F_z=(Dm)a_z \tag{8}$$

or

$$\sigma_{zz}^N Dx-\sigma_{zz}^S Dx+\tau^E Dz-\tau^W Dz-DW=(Dm)a_z \tag{9}$$

or

$$(\sigma_{zz}^N-\sigma_{zz}^S)Dx+(\tau^E-\tau^W)Dz-DW=(Dm)a_z \tag{10}$$

or

$$\frac{\partial \sigma_{zz}}{\partial z}DzDx+\frac{\partial \tau}{\partial x}DxDz-DW=(Dm)a_z \tag{11}$$

but

$$Dm=\rho DxDz \quad \text{and} \quad DW=g(Dm)=g\rho DxDz \tag{12}$$

and

$$a_z=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z} \tag{13}$$

Finally equation (11) leads to the momentum equation along z:

$$\boxed{\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right]=\frac{\partial \sigma_{zz}}{\partial z}+\frac{\partial \tau}{\partial x}-\rho g} \tag{14}$$

In addition, the continuity equation for incompressible fluids holds:

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0} \tag{15}$$

2 Newtonian fluids and the Navier-Stokes equations

For Newtonian fluids (fluids for most applications) the following **constitutive equations** apply:

$$\sigma_{xx} = -p+2\mu\frac{\partial u}{\partial x} \tag{16}$$

$$\sigma_{zz} = -p+2\mu\frac{\partial w}{\partial z} \tag{17}$$

$$\tau = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \tag{18}$$

where μ is the dynamic viscosity of the fluid, and p is the pressure.

By plugging equations (16), (17), and (18) for the stresses into the momentum equations (7) and (14), and by using the continuity equation (15) we get the **Navier-Stokes** equations:

$$\boxed{\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)} \tag{19}$$

and

$$\boxed{\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g} \tag{20}$$

The Navier-Stokes equations apply to both **laminar and turbulent** flows. However, in the case of turbulent flows their numerical solution requires very fine resolution in both time and space, and that requires significant computational resources and takes very long computing time. These methods are called DNS (Direct Numerical Simulations). However, a less laborious way of dealing with turbulent flows is provided from **averaging the flow-field in time**, as presented in the next section. Figure 2 shows a comparison flow visualizations of the instantaneous versus the average of a flow-filed around a sphere.

3 Reynolds averaging of the momentum equations, and the turbulent Reynolds stresses

In the case of turbulent flows we decompose the velocities u and w into their average (in time) values $\bar{u}$ and $\bar{w}$ and the **turbulent velocities** u' and w':

$$u = \bar{u}+u' \tag{21}$$
$$w = \bar{w}+w' \tag{22}$$

By taking the average in time of the momentum equations (7) and (14), as shown in Appendix A, the averaged momentum equations become:

$$\boxed{\rho\left[\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right]=\frac{\partial(\bar{\sigma}_{xx}-\rho\overline{u'u'})}{\partial x}+\frac{\partial(\bar{\tau}-\rho\overline{u'w'})}{\partial z}} \tag{23}$$

$$\boxed{\rho\left[\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right]=\frac{\partial(\bar{\sigma}_{zz}-\rho\overline{w'w'})}{\partial z}+\frac{\partial(\bar{\tau}-\rho\overline{u'w'})}{\partial x}-\rho g} \tag{24}$$

The three **Reynolds turbulent stresses** are then defined as follows:

$$\boxed{-\rho\overline{u'u'}=-\rho\overline{u'^2};-\rho\overline{w'w'}=-\rho\overline{w'^2};-\rho\overline{u'w'}} \tag{25}$$

In addition, the continuity equation also applies in the case of the mean velocities $\bar{u}$ and $\bar{w}$:

$$\boxed{\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{w}}{\partial z}=0} \tag{26}$$

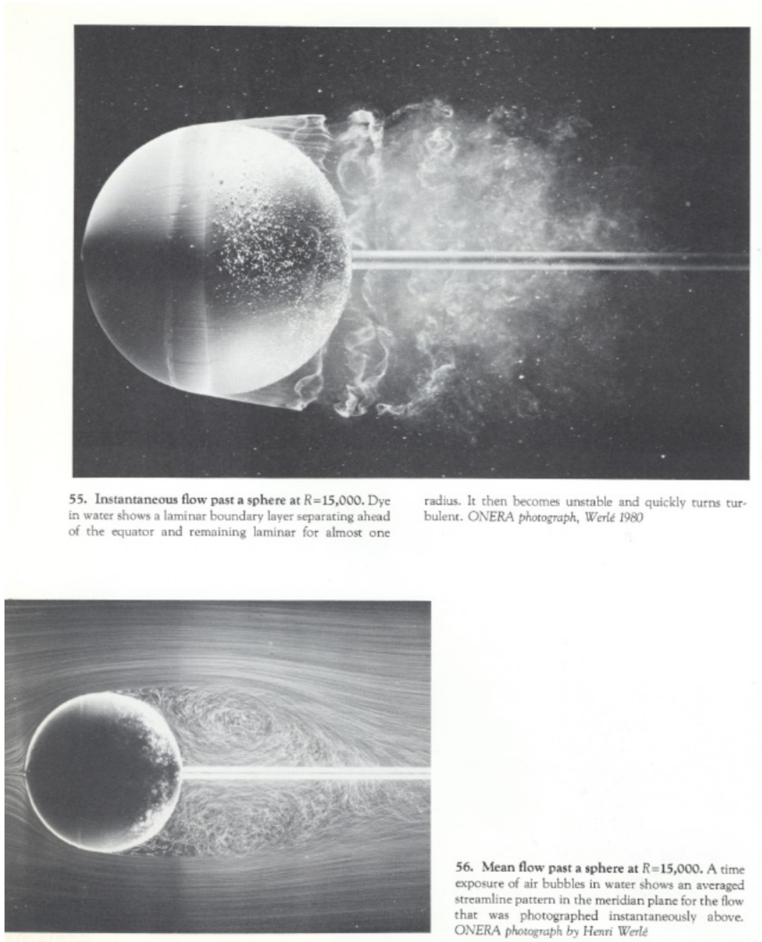

Figure 2: Views of instantaneous and average flow over a sphere. From Album of Fluid Motion.

4 Modeling of the Reynolds turbulent stresses

The Reynolds turbulent stresses can be modeled by using the Boussinesq approximation:

$$-\rho\overline{u'u'} = -\rho\overline{u'^2} = 2\mu_\tau \frac{\partial \bar{u}}{\partial x} - \rho k \quad (27)$$

$$-\rho\overline{w'w'} = -\rho\overline{w'^2} = 2\mu_\tau \frac{\partial \bar{w}}{\partial z} - \rho k \quad (28)$$

$$-\rho\overline{u'w'} = \mu_\tau \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) \quad (29)$$

where k is the **turbulence kinetic energy (TKE)**, which is associated with the kinetic energy of the turbulent velocities per unit mass, and is defined as follows:

$$k = \frac{1}{2}\left[\overline{u'^2} + \overline{w'^2}\right] \quad (30)$$

μ_τ is the **turbulent (dynamic) viscosity** or **eddy viscosity**, and can be determined from the following equation:

$$\mu_\tau = \rho C_\mu \frac{k^2}{\epsilon} \tag{31}$$

where $C_\mu = 0.09$ is a constant, and ϵ is the **turbulence dissipation rate** defined as follows:

$$\epsilon = \nu \left[\overline{\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}} + \overline{\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial x}} + \overline{\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial z}} + \overline{\frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z}} \right] \tag{32}$$

where $\nu = \frac{\mu}{\rho}$ is the **kinematic viscosity** of the fluid.

Note the similarity of equations (27), (28), and (29) with those of the viscous terms in the constitutive equations (16), (17), and (18).

k and ϵ are satisfying each their transport equations, which in the case of the very popular $k - \epsilon$ **model** can be found in *Turbulence Modeling for CFD by D.C. Wilcox (3rd ed., 2006)*, where many other **turbulence models** are also described. We do not include these equations here.

A Details of the Reynolds averaging of the momentum equations

In the case of turbulent flows the momentum equations, also provided again below, are averaged in time.

$$\rho \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right] = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau}{\partial z} \tag{33}$$

$$\rho \left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \right] = \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau}{\partial x} - \rho g \tag{34}$$

We define the average $\bar{A}$ of a quantity $A(x,z,t)$ over a time interval from t to $t + \Delta t$ as follows:

$$\bar{A} = \frac{1}{\Delta t} \int_t^{t+\Delta t} A dt \tag{35}$$

Note that the following expressions hold:

$$\overline{\frac{\partial A}{\partial x}} = \frac{1}{\Delta t} \int_t^{t+\Delta t} \frac{\partial A}{\partial x} dt = \frac{\partial}{\partial x} \left(\frac{1}{\Delta t} \int_t^{t+\Delta t} A dt \right) = \frac{\partial \bar{A}}{\partial x} \tag{36}$$

$$\overline{A + B} = \bar{A} + \bar{B} \tag{37}$$

$$\overline{AB} \neq \bar{A}\bar{B} \tag{38}$$

$$\overline{\bar{A}B} = \bar{A}\bar{B} \tag{39}$$

From the averaging of the continuity equation (15) we get:

$$\overline{\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}} = \overline{\frac{\partial u}{\partial x}} + \overline{\frac{\partial w}{\partial z}} = 0 \tag{40}$$

which finally leads to:

$$\boxed{\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0} \tag{41}$$

We also decompose the velocities u and w into their average value $\bar{u}$ and $\bar{w}$ and the **turbulent velocities** u' and w':

$$u = \bar{u} + u' \tag{42}$$
$$w = \bar{w} + w' \tag{43}$$

Note that, by their definition, the average values of the turbulent velocities are zero:

$$\overline{u'} = 0 \tag{44}$$
$$\overline{w'} = 0 \tag{45}$$

Then starting from equation (15) we get:

$$\frac{\partial(\bar{u} + u')}{\partial x} + \frac{\partial(\bar{w} + w')}{\partial z} = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{46}$$

and by using equation (26) we get:

$$\boxed{\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0} \tag{47}$$

So, from the equations above we see that both, the average velocities and the turbulent velocities, satisfy the continuity equation.

We will now take the average of the momentum equation (33) along x,

$$\overline{\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right]} = \overline{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau}{\partial z}} \tag{48}$$

$$\rho\left[\frac{\partial \bar{u}}{\partial t} + \overline{u\frac{\partial u}{\partial x}} + \overline{w\frac{\partial u}{\partial z}}\right] = \frac{\partial \bar{\sigma}_{xx}}{\partial x} + \frac{\partial \bar{\tau}}{\partial z} \tag{49}$$

We then take the averages of the following terms which appear on the left hand side of equation (49)

$$\begin{aligned}\overline{u\frac{\partial u}{\partial x}} = \overline{(\bar{u} + u')\frac{\partial(\bar{u} + u')}{\partial x}} &= \overline{\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial u'}{\partial x}} = \\ &= \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \overline{u'}}{\partial x} + \overline{u'}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}}\end{aligned} \tag{50}$$

$$
\begin{aligned}
\overline{w\frac{\partial u}{\partial z}} = \overline{(\bar{w}+w')\frac{\partial(\bar{u}+u')}{\partial z}} &= \overline{\bar{w}\frac{\partial \bar{u}}{\partial z} + \bar{w}\frac{\partial u'}{\partial z} + w'\frac{\partial \bar{u}}{\partial z} + w'\frac{\partial u'}{\partial z}} = \\
&= \bar{w}\frac{\partial \bar{u}}{\partial z} + \bar{w}\frac{\partial \overline{u'}}{\partial z} + \overline{w'}\frac{\partial \bar{u}}{\partial z} + \overline{w'\frac{\partial w'}{\partial z}}
\end{aligned}
\tag{51}
$$

and by using the fact that $\overline{u'} = 0$ and $\overline{w'} = 0$ the two equations above simplify to:

$$
\overline{u\frac{\partial u}{\partial x}} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}} \tag{52}
$$

$$
\overline{w\frac{\partial u}{\partial z}} = \bar{w}\frac{\partial \bar{u}}{\partial z} + \overline{w'\frac{\partial u'}{\partial z}} \tag{53}
$$

We then use the following identities for the derivatives of a product to get:

$$
\overline{u'\frac{\partial u'}{\partial x}} = \overline{\frac{\partial u'u'}{\partial x} - \frac{\partial u'}{\partial x}u'} = \frac{\partial \overline{u'u'}}{\partial x} - \overline{\frac{\partial u'}{\partial x}u'} \tag{54}
$$

$$
\overline{w'\frac{\partial u'}{\partial z}} = \overline{\frac{\partial w'u'}{\partial z} - \frac{\partial w'}{\partial z}u'} = \frac{\partial \overline{w'u'}}{\partial z} - \overline{\frac{\partial w'}{\partial z}u'} \tag{55}
$$

Plugging the expressions above into equations (52) and (53) we get:

$$
\overline{u\frac{\partial u}{\partial x}} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \frac{\partial \overline{u'u'}}{\partial x} - \overline{\frac{\partial u'}{\partial x}u'} \tag{56}
$$

$$
\overline{w\frac{\partial u}{\partial z}} = \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{w'u'}}{\partial z} - \overline{\frac{\partial w'}{\partial z}u'} \tag{57}
$$

Adding the two equations above we get:

$$
\overline{u\frac{\partial u}{\partial x}} + \overline{w\frac{\partial u}{\partial z}} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{w'u'}}{\partial z} - \overline{\frac{\partial u'}{\partial x}u'} - \overline{\frac{\partial w'}{\partial z}u'} \tag{58}
$$

Then by using equation (47) we get for the last two terms of the equation above:

$$
-\overline{\frac{\partial u'}{\partial x}u'} - \overline{\frac{\partial w'}{\partial z}u'} = -\overline{u'\left[\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right]} = 0 \tag{59}
$$

which renders

$$
\overline{u\frac{\partial u}{\partial x}} + \overline{w\frac{\partial u}{\partial z}} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{w'u'}}{\partial z} \tag{60}
$$

And by plugging equation (60) into equation (49) we get:

$$
\rho\left[\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{w'u'}}{\partial z}\right] = \frac{\partial \bar{\sigma}_{xx}}{\partial x} + \frac{\partial \bar{\tau}}{\partial z} \tag{61}
$$

which, with some rearrangement, leads to the averaged momentum equation along x

$$\boxed{\rho\left[\frac{\partial\bar{u}}{\partial t}+\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right]=\frac{\partial(\bar{\sigma}_{xx}-\rho\overline{u'u'})}{\partial x}+\frac{\partial(\bar{\tau}-\rho\overline{u'w'})}{\partial z}} \tag{62}$$

Similarly, it can be shown that the average of equation (34) leads to:

$$\boxed{\rho\left[\frac{\partial\bar{w}}{\partial t}+\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{w}\frac{\partial\bar{w}}{\partial z}\right]=\frac{\partial(\bar{\sigma}_{zz}-\rho\overline{w'w'})}{\partial z}+\frac{\partial(\bar{\tau}-\rho\overline{u'w'})}{\partial x}-\rho g} \tag{63}$$

The three **Reynolds turbulent stresses** are then defined as follows:

$$\boxed{-\rho\overline{u'u'}=-\rho\overline{u'^2};-\rho\overline{w'w'}=-\rho\overline{w'^2};-\rho\overline{u'w'}} \tag{64}$$