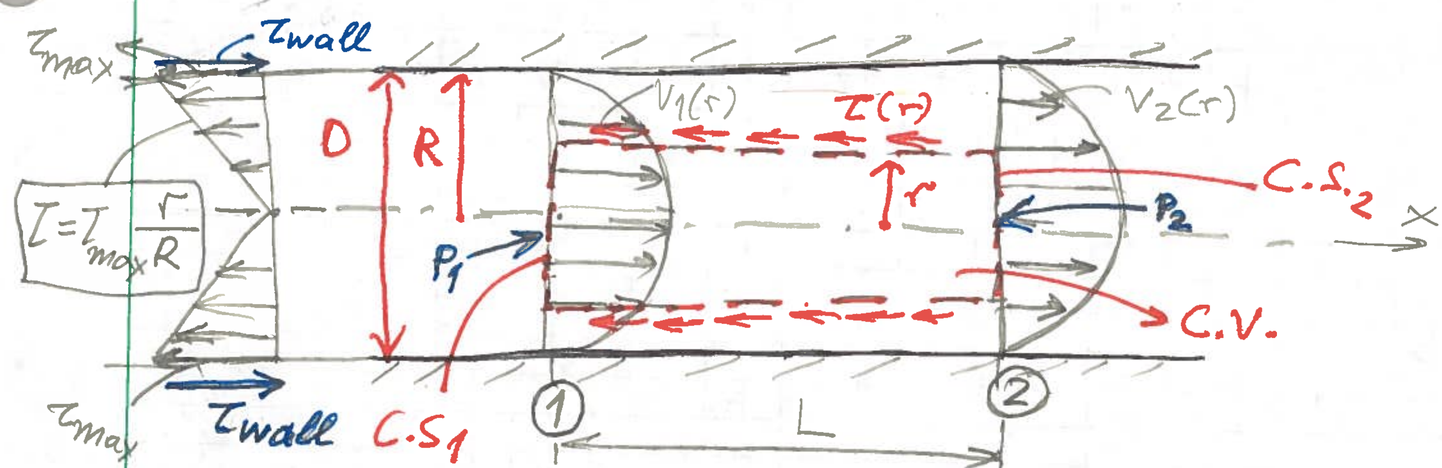
Developed Flows Inside Circular Pipes



Developed flow: Does NOT change along x.

Momentum equ. on shown C.V. (along x)

$$\sum F_x = \int_{C.S.} \rho V_x \vec{V}\cdot d\vec{A} \qquad \text{(steady flow)}$$

$$P_1 A_1 - P_2 A_2 - \tau(2\pi r \cdot L) = \underbrace{\int_{C.S_2} \rho V_2 V_2 dA - \int_{C.S_1} \rho V_1 V_1 dA}_{=0 \text{ since } V_2(r)=V_1(r)}$$

$$A_1 = A_2 = \pi r^2$$

$$\Rightarrow (P_1 - P_2)\cancel{\pi} r^{\cancel{2}} = \tau(2\cancel{\pi}\cancel{r}\cdot L) \Rightarrow \tau = \frac{P_1-P_2}{2L}\cdot r \quad (1) \quad \text{(shear stress on fluid)}$$

$$\underbrace{\tau_{max} = \tau_{wall}}_{\text{equal and opposite direction}} = \frac{P_1-P_2}{2L}\cdot R \quad (1')$$

$$\boxed{\tau = \tau_{max}\cdot\frac{r}{R} = \tau_{wall}\frac{r}{R}} \quad (2)$$

Valid for both LAMINAR & TURBULENT FLOWS

From (1') $\quad p_1 - p_2 = \tau_{max} \dfrac{2L}{R} \quad (3)$

Energy equ. between ① & ② $(h_p = 0, h_t = 0)$

$$\frac{p_1}{\gamma} + \cancel{z_1} + \cancel{\alpha_1 \frac{\bar{V}_1^2}{2g}} = \frac{p_2}{\gamma} + \cancel{z_2} + \cancel{\alpha_2 \frac{\bar{V}_2^2}{2g}} + h_L$$

$\alpha_1 = \alpha_2$ and $\bar{V}_1 = \bar{V}_2$ (WHY?)
(HINT: Developed Flow!)

Thus $\quad h_L = \dfrac{p_1 - p_2}{\gamma} \quad (4)$

(3) & (4) $\Rightarrow \quad h_L = \dfrac{\tau_{max}}{\gamma} \dfrac{2L}{R} = \dfrac{\tau_{max}}{\gamma} \dfrac{4L}{D} \quad (D = 2R)$

$$\boxed{h_L = \frac{\tau_{max}}{\gamma} \frac{4L}{D} = \frac{\tau_{wall}}{\gamma} \frac{4L}{D}} \quad (5)$$

NOTE: $h_L \sim \tau_{wall}$

(For inviscid flow $\tau_{wall} = 0 \Rightarrow h_L = 0$!)

Darcy-Weisbach equ: $\boxed{h_L = f \dfrac{L}{D} \cdot \dfrac{\bar{V}^2}{2g}} \quad (6)$

f → resistance coeff. or friction factor

$h_f \equiv h_L$
h_f → another symbol for pipes

Applies to both, LAMINAR & TURBULENT FLOWS

Combining (5) & (6)

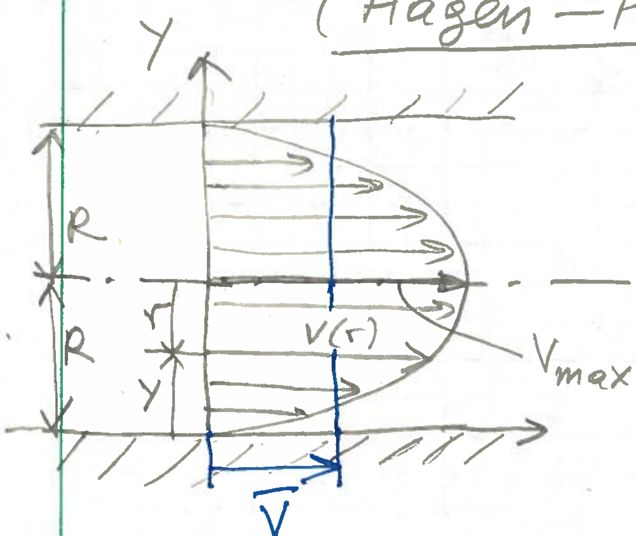
$$\frac{\tau_{max}}{\underbrace{\gamma}_{=\rho g}} \frac{4\cancel{L}}{\cancel{D}} = f \frac{\cancel{L}}{D} \frac{\bar{V}^2}{2\cancel{g}}$$

$$\boxed{\tau_{wall} = \tau_{max} = f \rho \frac{\bar{V}^2}{8}} \quad (7)$$

APPLIES TO BOTH LAMINAR & TURBULENT FLOWS!

LAMINAR FLOWS INSIDE CIRCULAR PIPES

(Hagen – Poiseuille flow)



μ: dynamic viscosity

$$\tau = \mu \frac{dv}{dy} \quad (y = R - r)$$

$$= \mu \frac{dv}{dr}\frac{dr}{dy} = -\mu \frac{dv}{dr}$$

$$\left(r = R - y \rightsquigarrow \frac{dr}{dy} = -1\right)$$

Thus $\tau = -\mu \dfrac{dV}{dr}$ (8)

But from (1) $\tau = \left(\dfrac{P_1 - P_2}{2L}\right) \cdot r = B\, r$ (9)

DEF. (9)

Then $B\, r = -\mu \dfrac{dV}{dr}$ (10)

$$Br = -\mu \frac{dV}{dr} \quad (10)$$

$$\rightsquigarrow \quad dV = -\frac{B}{\mu} r\,dr \rightsquigarrow \int_r^R dV = -\frac{B}{\mu}\int_r^R r\,dr \rightsquigarrow$$

$$\rightsquigarrow \quad V\Big|_r^R = -\frac{B}{\mu}\frac{r^2}{2}\Big|_r^R \rightsquigarrow \underbrace{V(R)}_{\substack{\shortparallel \\ 0}} - V(r) = -\frac{B}{\mu}\left[\frac{R^2}{2}-\frac{r^2}{2}\right]$$

0 (NO SLIP CONDITION!)

$$\rightsquigarrow \quad -V(r) = -\frac{BR^2}{2\mu}\left[1-\left(\frac{r}{R}\right)^2\right] \rightsquigarrow V(r) = \frac{BR^2}{2\mu}\left[1-\left(\frac{r}{R}\right)^2\right]$$

$$\rightsquigarrow \quad \boxed{V(r) = V_{max}\left[1-\left(\frac{r}{R}\right)^2\right]} \quad (11)$$

Parabolic velocity profile for laminar flows inside pipes

where: $\boxed{V_{max} = \frac{BR^2}{2\mu}}$ (12)

$$\bar{V}\,(\text{mean velocity}) = \frac{Q}{A} = \frac{\int_A v\,dA}{A} = \frac{V_{max}}{2}$$

using $V(r)$ from (11)

$$\boxed{\bar{V} = \frac{V_{max}}{2}} \quad (13)$$

← For LAMINAR flows inside pipes

From (9) $\underline{\tau_{max} = BR}$ (14)

Then from: $\tau_{max} = f \rho \frac{\bar{V}^2}{8}$ (7) and (14)

we get: $f \rho \frac{\bar{V}^2}{8} = BR$ (15)

But from (13) & (12): $\bar{V} = \frac{V_{max}}{2} = \frac{BR^2}{4\mu}$ (16)

$(15) \Rightarrow f \rho \frac{\bar{V}}{8} \bar{V} = BR$

or $f \rho \frac{\bar{V}}{8} \frac{\cancel{B} R^{\cancel{2}}}{4\mu} = \cancel{B}\cancel{R}$ or $f \rho \frac{\bar{V} R}{32\mu} = 1$

or $f \frac{\bar{V} \cdot \frac{D}{2}}{32 \underbrace{\left(\frac{\mu}{\rho}\right)}_{= \nu \text{ (Kinematic Viscosity)}}} = 1$ or $f \frac{\bar{V} D}{\nu} \frac{1}{64} = 1$

or $f \cdot \frac{Re}{64} = 1$ or $\boxed{f = \frac{64}{Re}}$ (17) ← FOR LAMINAR FLOWS IN PIPES

$\boxed{Re \underset{\text{DEF.}}{=} \text{Reynolds \#} = \frac{\bar{V} D}{\nu}}$ (18) ← dimension-less number

Re_D or R other symbols

very important number characterizing effects of viscosity in pipe flows!

(remember units of ν is $\frac{m^2}{s}$ in S.I.)

For Laminar flows
equ. (6) can also be written as:

$$h_L = f\,\frac{L}{D}\,\frac{\bar{V}^2}{2g} = \frac{64}{Re}\cdot\frac{L}{D}\cdot\frac{\bar{V}^2}{2g} =$$

$$= \frac{\cancel{64}^{\,32}}{\left(\frac{\cancel{\bar{V}}D}{\nu}\right)}\cdot\frac{L}{D}\cdot\frac{\bar{V}^{\cancel{2}}}{\cancel{2}g} = \frac{32\nu}{D}\,\frac{L}{D}\,\frac{\bar{V}}{g} =$$

$$= \frac{32\mu}{\rho}\,\frac{L}{D^2}\,\frac{\bar{V}}{g} \Longrightarrow$$

$$\boxed{h_f \equiv h_L = \frac{32\mu L\bar{V}}{\gamma D^2}} \quad (19)$$

$\leftarrow$ For LAMINAR FLOWS INSIDE PIPES

For TURBULENT FLOWS INSIDE PIPES:

Darcy-Weisbach equ. still applies

$$h_f \equiv h_L = f\,\frac{L}{D}\,\frac{\bar{V}^2}{2g} \quad (6)$$

where f is given from Prandtl equ.

$$\boxed{\frac{1}{\sqrt{f}} = 2\log_{10}\left(Re\sqrt{f}\right) - 0.8} \quad (20)$$

FOR TURBULENT FLOWS INSIDE SMOOTH PIPES

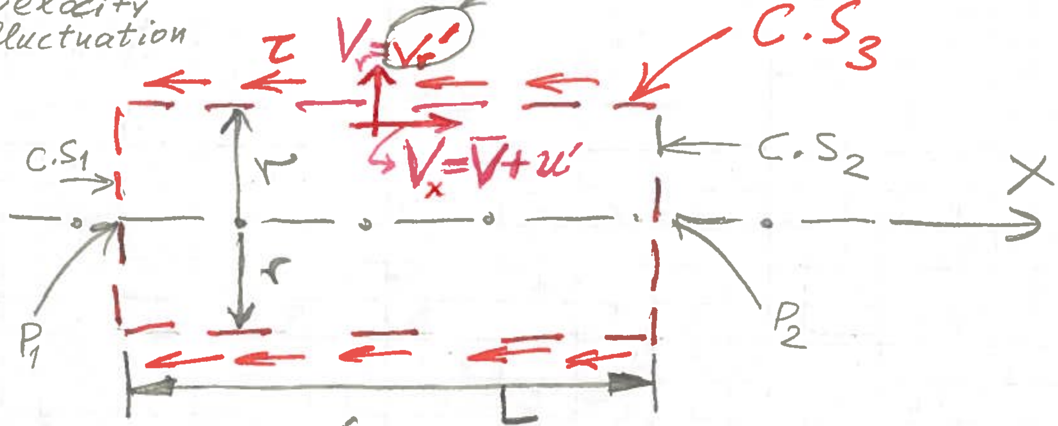
For Rough pipes (or smooth) $\Rightarrow$ $\boxed{\text{Moody Chart (or Diagram)!}}$

In the case of turbulent flows:

$V_x = \bar{V} + u'$ → turbulent axial velocity fluctuation

$\bar{V}$: time averaged axial velocity

turbulent radial velocity fluctuation



Note:

$$\boxed{\begin{aligned}\overline{u'} &= 0\\ \overline{V_r'} &= 0\end{aligned}}$$

Momentum equ.: $\sum F_x = \int_{C.S._3} \rho V_x V_r \, dA$ ← no longer zero!

(For Laminar flows the radial velocity $V_r = 0$)

Note: See equ. (25d) on next page regarding $\int_{C.S_1}$ & $\int_{C.S_2}$

$$P_1 A_1 - P_2 A_2 - \tau (2\pi r) \cdot L = \int_{C.S._3} \rho (\bar{V} + u') v_r' \, dA \quad (21)$$

due to equ. (8)

$$(P_1 - P_2)\pi r^2 + \mu \frac{dV}{dr}(2\pi r)L = \int_{C.S_3} \rho (\bar{V} + u') V_r' \, dA \quad (22)$$

Note: $p = \bar{p} + p'$ ← turbulent fluctuation: $\overline{p'} = 0$

($\bar{p}$: time averaged)

after averaging in time

$$(\bar{P}_1 - \bar{P}_2)\pi r^2 + \mu \frac{d\bar{V}}{dr}(2\pi r)L = \int_{C.S_3} \rho \, \overline{u' V_r'} \, dA \quad (23)$$

$$(\bar{P}_1 - \bar{P}_2)\pi r^2 + \mu \frac{d\bar{V}}{dr}(2\pi r)L = \rho \overline{u' v_r'}\,(2\pi r)L \quad (24)$$

Assumptions we have made to get to equ. (24)

- $\overline{u' v_r'}$ only depends on r (25a)
- $\overline{\bar{V} v_r'} = \bar{V}\,\overline{v_r'} = \bar{V}\cdot 0 = 0$ (proved, since $\overline{v_r'} = 0$) (25b)
- $\frac{d}{dt}\int_{C.V.} \rho(\overline{\vec{q}})\,d\forall = 0$ (25c) (can be proved) (with $\vec{q} = (\bar{V}+u')\vec{i} + v_r'\vec{j}$)
- $\int_{C.S_1} \rho(\bar{V}+u')^2\,dA = \int_{C.S_2} \rho(\bar{V}+u')^2\,dA$ (can be proved) (25d)

We finally get:

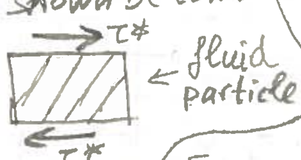
$$(\bar{P}_1 - \bar{P}_2)\pi r^2 + \underbrace{\left[\mu \frac{d\bar{V}}{dr} - \rho\overline{u' v_r'}\right]}_{\text{New } \tau^*}(2\pi r)L = 0 \quad (24a)$$

$$\boxed{\tau^* = \mu \frac{d\bar{V}}{dr}\left(-\rho\overline{u' v_r'}\right)} \quad (26)$$

Reynolds turbulent stress!

Note 1: $\boxed{\tau^* = -\tau}$

Actually τ^* is positive in the direction shown below:



Note 2: $\frac{d\bar{V}}{dr} < 0$

$$\Rightarrow \quad -\left[\mu \frac{d\bar{V}}{dr} - \rho\overline{u' v_r'}\right] = \frac{\bar{P}_1 - \bar{P}_2}{2L}\, r \quad \text{or} \quad \boxed{-\tau^* = \frac{\bar{P}_1 - \bar{P}_2}{2L}\, r} \quad (27)$$

Compare equ. (27) with equ. (1) (remember $\tau^* = -\tau$)